RADIO COMMUNICATION

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THEORY AND METHODS

WITH AN APPENDIX

ON

TRANSMISSION OVER WIRES

BY

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PREFACE

This book is the substance of a course of lectures given by the author during the summer of 1917 to a Company of the U. S. Reserve Signal Corps troops. From the employees of the Western Electric Company who had volunteered for service two companies were formed, one at Hawthorne, Ills., and the other at New York City. These men were retained on the pay roll of the Company and their training was carried out on company time by other employees. In radio-communication this training consisted in part of the course of lectures mentioned above and in part of laboratory work. To Major Jewett, Chief Engineer of the Western Electric Company, the author is indebted for this opportunity of assisting in the training of our Expeditionary Forces. It is in the hope that the method of presentation, found acceptable to this Signal Corps Company, may be of wider use that this text is published.

The individual men to whom the lectures were given differed widely in the extent of their previous training in electrical engineering. The author, therefore, adopted a method which involved practically no mathematics except elementary algebra and presupposed but a limited knowledge of physics. (This method solves problems like that of determining the natural frequencies of tuned and coupled circuits or that of finding the effective value of a sinusoid.) Described in mathematical terms it consists in the development and use of two concepts, namely, that of the vector operator "j" and the differential operator "p." Since all the functions with which the radio-engineer has to deal are expressible in exponential form by the use of "j," only one very special case of the operator "p" need be considered. With a knowledge of this one case, which may be developed in an elementary manner, the student becomes equipped with tools of analysis which in some school curricula are delivered to him only through a two years' drill in calculus, and differential equations.

In selecting the material for these lectures there was eliminated all that of purely historical interest, *e.g.*, coherers, and also that of purely theoretical interest, *e.g.*, much of the conventional text-book material on the transmission of electromagnetic waves. The aim throughout has been to present fundamental principles and methods rather than detailed instruction as to apparatus and its operation. The present rapid development of radio-apparatus makes such descriptive material a necessary accompaniment of the manufacturer's apparatus rather than a desirable part of a text. The student trained in the fundamentals will find small difficulty in dealing with the apparatus of that new stage of the art into which radio-communication seems to be entering.

In order that as many as possible of these fundamental principles shall be available to the reader whose working knowledge even of algebra is negligible, the material involving algebra has been concentrated into chapters *I* and *V*. The important chapters, *III*, *IV*, *VI* and *VII*, on the vacuum tube, on detection, on undamped waves, and on radio-telegraphy and telephony have thus been made practically non-mathematical. Since the method followed in chapters *I* and *V* is readily applicable to the more difficult problems of transmission over wires an appendix is added in which such application is made.

In the diagrams accompanying the text the symbols standardized by the I.R.E. have been used consistently. But few footnote references have been introduced and those mostly to the Proceedings of the Institute of Radio Engineers. Of the papers presented to this Institute in the past few years many have been the first disclosures of important advances in the art. For permission to reproduce some of the illustrations of these papers the author expresses his thanks to the Institute. Thanks are also expressed to Mr. C. R. Englund for reading part of the proof.

JOHN MILLS.

NEW YORK,
October, 1917.

A NOTE FOR THE NON-MATHEMATICAL READER

Each science or art has its own peculiar terminology. To the wireless engineer, a "tikker" is a definite piece of apparatus. To those untrained in wireless, the word might seem to represent anything from a clock to a stock market printing telegraph. Drivers of oxen used the words "gee" and "haw" to operate upon a team of oxen so as to change its direction 90° to the right or to the left. An officer operates upon a squad to rotate the direction in which it is facing 90° to the left by the command "left face." The mathematician operates upon a line pointing in any given direction to rotate it 90° counter-clockwise (*i.e.*, left handed) with the symbol "j." If the sergeant operates upon his squad twice in succession with the command "left face" the result is an "about face" or a complete reversal of direction. If the mathematician operates twice in succession with "j" he too obtains a complete reversal which he represents by a minus sign ($-$). This idea of representing a 90° rotation by the symbol "j" is inherently no more difficult than the idea of "left face." Its use is fundamental to the theoretical portions of this text.

Another idea fundamental to the text is that of a "rate of change." A speedometer operates to tell the driver *at any instant* the speed or rate of change of position of his car. In our daily life there are many similar illustrations of instantaneous speeds or rates of change. Now if the mathematician wishes to represent the operation of finding at any instant the rate of change of some quantity, he does so by prefixing to it the letter 'p." Thus if S represents the space (*e.g.*, miles) the car travels then pS will represent its speed. The use of a symbol like "p" to represent an operation, namely "the operation of finding the rate of change for the quantity to which p is prefixed" should be con-

sidered of the same grade of difficulty as the use of symbols composed of dashes and dots to represent letters and words in the familiar Continental Code.

Beyond these two fundamental ideas, involved in the symbols "j" and "p," there is no mathematics in this text except simple algebra, that is the addition, subtraction, multiplication, and division of quantities expressed by letters, instead of by numbers as in arithmetic. Because the quantities are electrical, *e.g.*, current, voltage, and resistance, the letters used are i, v, R, etc. instead of the usual a, b, c, and x, y, z of algebra.

Although the ideas involved are simple, as is seen above, the ready use of these ideas in their symbolic expression requires practice just as the ready use of the symbols of the telegraph code requires practice. To assist the student in such practice, some problems of gradually increasing difficulty are given with solutions or answers in Part I of the problems at the end of the text.

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CHAPTER I

ALTERNATING CURRENTS

Electricity.—In the world in which we live we recognize the existence of ponderable matter from its characteristic of inertia, that is, from the fact as Newton expressed it, that "any body continues in a state of rest or of uniform motion in a straight line, except insofar as compelled by force to change that state." It is by the forces we exert, the reactions they meet, and the motions they produce that we obtain our earliest and most fundamental ideas of this universe. By observations of the motions produced, the early scientists discovered the phenomena of electrification. Later observations and careful reasoning showed the divisibility of matter into molecules and of molecules into atoms. Still more recent work of scientists has demonstrated the existence of subdivisions of the atom, which have been called electrons. While molecules and atoms have specific properties depending upon the substance from which they are obtained, the characteristics of an electron are independent of the source, that is, chemical substance or material, from which it is derived.

In the earlier development of the science of electricity it was recognized that all electrified or charged bodies can be grouped into two classes, such that all the bodies of either class repel all the other bodies of the same class, but attract all bodies of the other class. There thus appeared to be two kinds of electrification, and the terms "positive" and "negative" were arbitrarily

introduced for purposes of distinction between the two classes. This terminology is used today, and however charges of electricity are produced, it is agreed by scientists to call a charge positive when it is repelled by a glass rod which has been rubbed with silk and negative when it is repelled by an ebonite rod which has been rubbed with cat's fur.

According to this terminology the electron, which is a charged particle or corpuscle, is negatively charged. It is more exact to say that the electron is a negative charge, for it has no recognizable mass in the same sense as ponderable matter has mass. It exhibits inertia when subjected to electrical repulsion by other electrons, but the amount of inertia depends upon the velocity with which it is moving. It is convenient, however, for purposes of forming a concept of the electron, to consider it to have an inertia or mass about 1/1860 part of that of the atom of hydrogen.

In the structure of matter electrons are associated with nuclei of atomic size and form atoms which show no properties of electrical charges. An atom (or molecule) may have dissociated from it by various means one or more of its component electrons. When an atom has lost an electron it has, as viewed from the standpoint of the normal uncharged atom, a deficiency of negative electricity and therefore has the characteristics of a positive charge. The phenomenon of charging a body consists then in either adding electrons to it or subtracting them from it.

The conduction of electricity is accomplished by the motion of carriers of electricity, the nature of which is determined in part by the medium. In the case of solid conductors the electrons are the carriers. In the case of liquid and aeriform media, in addition to the electrons, the molecules or atoms are free to move and when charged these also act as carriers of electricity. The carriers obey the law that like charges repel and unlike attract, and their motions are due to such attractions or repulsions.

Electrical Magnitudes.—In the conduction of electricity we distinguish three magnitudes: namely, the total quantity of electricity transferred; the rate at which electricity is transferred,

called the current; and the cause of the transfer, called the electromotive force. For a given electromotive force the current will depend upon the nature of the conducting path through which the transfer is effected. The ratio of the electromotive force (cause) to the current (effect) is called the impedance of the conducting path.

Electrical Units.—To express numerically these magnitudes of an electrical circuit, namely, electromotive force, current and impedance, requires the adoption of a system of units in which they may be measured. Just as various units have been adopted for the expression of non-electrical magnitudes—as for example, length and area—so various units, or rather systems of units, have been used in the science of electricity. Of the three systems in use, one, the so-called practical system, takes account but indirectly of the physical relations involved and defines the units in terms of certain standards or methods of measurement. In this system the units of electromotive force, current, and impedance are respectively the volt, the ampere, and the ohm. Thus a steady e.m.f. of 1 volt acting in a circuit of impedance 1 ohm causes a steady current of 1 ampere to flow. A current of 1 ampere flowing for 1 second results in a transfer of a quantity of electricity called a coulomb.

Resistance.—The current which flows in a circuit depends both upon the e.m.f. and upon the nature of the circuit. The ratio at any instant of the e.m.f. to the resulting current is the instantaneous value of the impedance. In a circuit where no storage of electrical energy takes place, that is, a circuit containing no condensers or inductances, the instantaneous value of the impedance is independent of the previous electrical history of the circuit. In circuits involving energy storage, the transfer of electricity going on at any instant is due not only to the impressed e.m.f. but also to the redistribution of energy occasioned by the storage reservoirs. In the former type of circuit, energy is dissipated but not stored, and the dissipated energy is supplied directly and instantaneously by the impressed e.m.f. While for

this case the instantaneous impedance is independent of the previous history of the circuit, it is not necessarily a constant independent of the current. For example, doubling the e.m.f. applied to a circuit may or may not result in double the current depending upon the nature of the conducting circuit. Where the conducting path is metallic (and essentially free from inductance) the current is proportional to the e.m.f. if the other conditions such as temperature remain constant. In this case the impedance is constant, and Ohm's law holds. The impedance is then spoken of as an ohmic resistance. On the other hand, in the cases of conduction through a vacuum or through a gaseous medium, Ohm's law does not hold. The impedance of a gaseous path is in general a pure resistance, but the ratio of e.m.f. to current is not a constant. Fig. 1 shows typical e.m.f.-current curves for the two cases.

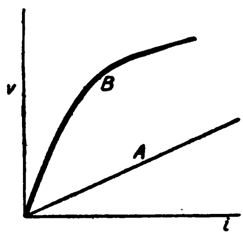


FIG. 1.—E.m.f.-current characteristics of resistance.

In curve A is represented the case of a metallic conductor, for which

$$v = Ri$$

where R is the impedance (resistance) of the circuit.

Or we may write:

$$i = v/R = kv$$

in which case k is the conductivity of the circuit.

In the case of the gaseous conducting path, the relations of which are as shown in curve B, we may write:

$$i = a + bv + cv^2 + dv^3$$

where a, b, c, and d are constants.

Storage Reservoirs for Electrical Energy.—Electrical energy may be stored either in a magnetic field or in an electrostatic field. The magnetic field may be created by a current flowing in a conducting circuit, in which case the circuit is called induc-

tive. The magnetic field, and hence the energy stored in it, will depend upon the geometrical form of the circuit and upon the current flowing in the circuit. The factor by which the energy for any given circuit is determined from a knowledge of the current is called the inductance of the circuit. If the path followed by the magnetic lines of force is free from iron or other magnetizable substances the inductance is a constant independent of the current. If iron is present in the magnetic field formed by the current in the circuit, the inductance is in general much greater than for air, and, because of the variations in permeability of the iron, is not constant but is a more or less complicated function of the current. For the wireless engineer most of the inductances used are essentially iron-free.

Inductance.—It is convenient to define the inductance of a circuit in terms of the current and the e.m.f., instead of in terms of current and energy as implied above or, as is frequently done, in terms of current and magnetic flux. Whenever a change occurs in the magnetic field linking with an electrical conductor there is induced in the conductor an e.m.f. This e.m.f. of self-induction is, of course, equal and opposite to the e.m.f. which would have to be applied to the conductor to produce the same change in magnetic field. The change in magnetic field is produced by a change in the current flowing in the circuit, and the inductance or self-inductance of a circuit is defined as the ratio between the e.m.f. induced and the rate at which the current changes in value. Thus let v represent the value of the e.m.f., L the self-inductance, and i the value of the current; and also let p stand for the expression "rate of change of," so that pi means "the rate of change of i." Then

$$L = v/pi \text{ or } v = Lpi \tag{1}$$

A circuit will then have unit inductance when 1 unit of e.m.f. is required to maintain a current which is changing at unit rate; that is, in the practical system a circuit has unit inductance,

namely, 1 henry, when an e.m.f. of 1 volt is required to produce a change in current at the rate of 1 ampere per second.

Capacity.—Energy is also stored in the electrical field existing between two or more charged bodies. In this case the field, and hence the energy, depends upon the charges or quantities of electricity and upon the geometrical configuration of these charges. A system of conductors insulated from one another and so capable of holding charges and of maintaining a field between the conductors is called a condenser. The capacity of a condenser is that factor, by the use of which the energy may be calculated from a knowledge of the charges. Between the conductors or plates of a condenser there exists, of course, an e.m.f. available in case a conducting path is supplied from one plate to the other for producing a current. The capacity may be defined also in terms of this e.m.f. and the current. This e.m.f. is obviously equal and opposite to that which would have to act in such a conducting circuit in order to bring about this distribution of charges. In terms of this e.m.f. the capacity is defined as the ratio of the charging current flowing to the plates and the rate of variation of the e.m.f. causing this current. Hence if i represents the charging current and v the e.m.f., the capacity is defined by

$$C = i/pv \tag{2}$$

where p has the same significance as before. The unit of capacity is then that of a condenser, where a unit charging current will be maintained by increasing the e.m.f. at unit rate. Or, in the practical system, a farad is the capacity of a condenser where a charging current of 1 ampere is maintained by an e.m.f. increasing at the rate of 1 volt per second.

The Operator p.—The letter p, meaning "rate of change of," is an operator in the same way as the letters log and sin are operators, indicating the results of performing the operation of taking the logarithm or the sine of the quantity to which they may be prefixed. Also, just as $\log^{-1}$ and $\sin^{-1}$ represent results the inverse of these operations, namely, the operation of find-

ing the quantity whose logarithm or whose sine is the quantity to which they may be prefixed, so p^{-1} is an inverse operator representing the result of finding the quantity whose rate of change is that of the quantity to which the operator is prefixed. Thus if

$$x = \sin y,$$

then

$$\sin^{-1} (\sin y) = y,$$

or

$$\sin^{-1} x = y.$$

And if

$$i = pv,$$

then

$$p^{-1}i = p^{-1} pv = p^0 v = v$$

or if

$$i = Cpv,$$

where C is a constant and does not vary with v or i, then

$$p^{-1}i = p^{-1} (Cpv) = Cp^{-1} pv = Cv,$$

or

$$v = p^{-1} i/C \tag{3}$$

Impedance.—As we have seen above, it is only in the case of a circuit formed by a pure resistance that we can state the numerical value of the impedance without a knowledge of either the applied e.m.f. or the resulting current. Considering the case of a circuit formed by an inductance, we have in equation (1) an expression of the fact that the value of the e.m.f. required at any instant to force a current i through the inductance depends upon the rate at which the current is changing in value at that instant. Obviously if the current is constant its rate of change is zero, and no e.m.f. is required to overcome the self-inductance of the circuit. If the current is increasing, its rate of change is positive. On the other hand, if the current is decreasing its rate, pi, is negative and the e.m.f. required is

negative. In other words, for this case the required e.m.f. is less than zero; that is, the inductance by its own properties tends to maintain a current.

What impedance a given inductive circuit offers depends then upon the rate of change of current within the circuit. The cases of most interest to the engineer are those in which the current is periodic; that is, undergoes similar variations at times separated by constant intervals or periods. The complete series of values assumed by the current in the period is called a cycle, and the number of cycles per second, that is, the reciprocal of the periodic interval, is called the frequency. Such a current may be either pulsating or alternating. In the former case the direction of transfer of electrons along the conducting circuit remains always the same, but the number of electrons transferred per second, that is, the current, varies as time progresses. In the latter case both the value of the current and its direction in the circuit undergo variations with time. As will be seen later, a pulsating current may be considered to be the resultant of a steady unidirectional current and a superimposed alternating current.

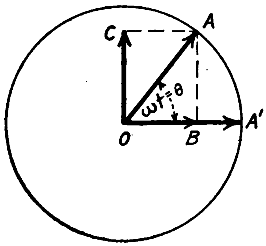


FIG. 2.—Definition of sin θ and cos θ.

Sinusoidal Alternating Functions.—The simplest case of an alternating current or an alternating e.m.f. is one in which the successive values assumed by the current are proportional to the sine of an angle, which in turn increases constantly as time progresses. Such functions are called sinusoidal. The definition of the sine of an angle as the ratio in a right-angled triangle constructed on the angle of the side opposite the angle to the hypothenuse is a special definition limited to angles less than 90°. In general the sine of an angle θ is to be found by assuming a definite length of line, *e.g.*, *OA* in Fig. 2, to be rotating about one extremity *O*, so that the other extremity *A* describes a

circle, by defining the angle θ as the amount of rotation from some assumed position as OA', and then by defining the sine of this angle as the ratio of the length AB of the perpendicular from A upon OA' to the length of the rotating line; that is

$$\sin \theta = AB/OA.$$

If the rotating radius OA is taken of unit length, then

$$\sin \theta = AB.$$

If AB is above the line OA' it is taken as positive, and if below as negative. The angle θ is positive if measured counter-clockwise from OA'. If the radius OA rotates through 360° or 2π radians in a time T, then its angular velocity is $2\pi/T$, and if time is measured from the instant that OA is coincident with OA',
then

$$\theta = 2\pi t/T,$$

where t represents the time.

It is sometimes convenient to show the successive values assumed by the sine by plotting them against an axis of time as

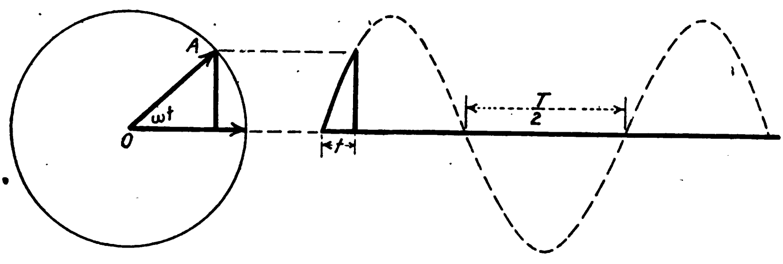


FIG. 3.—A sinusoidal function.

shown in Fig. 3. A sinusoidal function is then of the form $v = E \sin 2\pi t/T$. At $t = 0$ this function is zero. It has a maximum value of E at $t = T/4$. When $t = T/2$ it is again zero. From $t = T/2$ to $t = T$ it is negative, reaching its minimum value of $-E$ at $t = 3T/4$. The time T is then the periodic

interval required for the function to pass through its cycle of values. The frequency f is, as above, $1/T$. Hence

$$v = E \sin 2\pi ft$$

or for convenience,

$$v = E \sin \omega t$$

The equivalents ω, $2\pi f$ and $2\pi/T$ are all expressions for the angular velocity of a rotating radius, to the projection of which upon some reference line the instantaneous values of the sinusoidal function are proportional.

Referring again to Fig. 2, the cosine of θ is defined by

$$\cos \theta = OB/OA, \text{ or } \cos \theta = OB, \text{ when } OA = 1.$$

It is also evident that

$$\cos \theta = \sin (\theta + 90°)$$
$$\cos \theta = - \sin (\theta - 90°)$$

Vectors.—It is convenient to consider the rotating radius of Fig. 2 as but a special case of a vector. A vector is defined as a quantity having both magnitude and direction. With a vector representation of displacements, velocities, and forces, the student of elementary mechanics is already familiar. The law for the addition of two vectors is most easily seen when the vectors represent motions. Thus let the point m of Fig. 4 undergo two displacements represented in direction and magnitude by the vectors a and b. Obviously, due to displacement a the point m must lie somewhere on the line PQ, and due to displacement b, somewhere on the line RS, and hence at m'. The resultant displacement is c, which is a vector representing the combined effects or addition of a and b. This resultant is seen to be the diagonal of a parallelogram constructed on the two vectors a and b as sides. The rule for the addition of vectors may be better expressed as follows: From the extremity of one vector, as a, draw the other vector, as b, in its proper direction. The vector c required to complete the triangle is the resultant.

Conversely any vector, as c, may be assumed to be the resultant of any two vectors which, when combined as above, will give c for their resultant. Thus in Fig. 5, c is the resultant of a' and b' and also of a'' and b'', or it may be the resultant of an infinite number of combinations other than those shown in the figure. The vectors of each pair, as a' and b', are called the components of c, and the process of finding a pair is called resolution. In general the resolution of a vector c is made into rectangular components, as a''' and b''' of Fig. 6, one of which lies along some desired axis. In this case

$$a''' = c \cos \theta \quad \text{and } b''' = c \sin \theta$$

Returning now to a consideration of the rotating vector OA which was used to define the sine of an angle, it is seen in Fig. 2

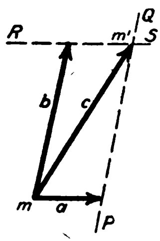


FIG. 4.—Addition of vectors.

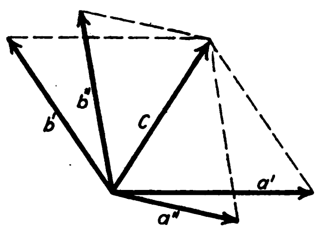


FIG. 5.—Resolution of a vector.

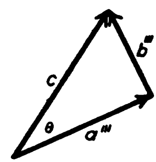


FIG. 6.—Rectangular components of a vector.

that it may be resolved into two components, as OB and BA or OB and OC. Let OA be of unit length; then the vector OB is defined as the cosine and the vector OC as the sine of the angle. For all angles between 0 and 180° the vector OC points upward and varies only in magnitude, not in direction, but for angles between 180° and 360° it points downward. Hence if we take the upward direction as positive, we should call the reverse direction negative. Thus the vector OC, defining the sine, gives at once not only the numerical value but also the algebraic sign. Similar reasoning shows that the sign of the cosine is given by the vector OB.

Vector Operators.—We have just made use of the idea that reversing the direction of a vector means reversing its algebraic sign. Now reversing a vector may be considered to mean rotating it through 180°. Given a vector X, then we can indicate the result of the operation of rotating it through 180° by prefixing to X a minus sign, or writing $(-1)X$. We may then look upon (-1) as an operator, which when prefixed to a vector represents the result of reversing it or of rotating it through 180°.

Now it is convenient to represent the result of other operations upon directed quantities, as for example, the operation of rotating through 90°. To represent such an operation the symbol j is used. Thus jX represents a rotation of the vector X through 90° counter-clockwise. If we operate upon this new vector jX by the operator (-1), we get $-jX$, which is a vector of length that of X but rotated through $90° + 180°$ or $-90°$ from the original position. Thus for illustration, if we are dealing with the points of a compass and start from the direction east, say E, then north is jE, west is $-E$ and south is $-jE$, as in Fig. 7.

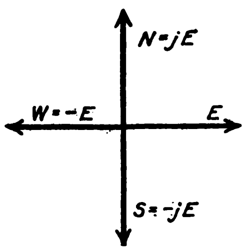


FIG. 7.—The operator "j."

A rotation of 360° may be considered to be the result either of an operation of rotating through 360°, thus leaving the vector in its original position, or as the result of two successive operations, each rotating through 180°. Thus $-X$ represents a rotating of X by 180° counter-clockwise, and $-(-X)$ represents a rotation of $-X$ by 180°, that is, a rotation of X by 360°, bringing it back to its original position. Then $-(-X)$ must equal X, as it does. Similarly a rotation of 180° may be thought of as two successive rotations of 90° each. Thus jX represents a rotation of X by 90° counter-clockwise, and jjX represents a rotation of jX by 90°, or of X by 180°. But $-X$ also represents a rotation of X by 180°. Hence $-X = jjX = j^2X$, if we let the

exponent represent the number of successive times the operator j is applied. Then $-1\ (X) = j^2(X)$, or $-1 = j^2$ and j is said to be $\sqrt{-1}$, which is an imaginary quantity, the square root of minus one. For convenience vectors operated upon by j are called imaginaries and those not so operated upon are called reals.

Returning again to Fig. 2, the rotating vector OA in any position, as that shown, is the sum of the vectors OB and OC. Let us take as a reference direction that of the line OB. Now OB is $\cos\theta$ and OC is $\sin\theta$, but the direction of OC is 90° counter-clockwise from the assumed reference direction. The vector OC may therefore be considered to be the result of laying off along the reference direction a vector of length OC and then of operating to rotate it through 90°. Thus OC may be written as $j\sin\theta$. The rotating vector OA is then the sum of the two vectors $\cos\theta$ and $j\sin\theta$.

The vector OA is, however, to be considered as the result of operating upon a vector as OA' of the same length or magnitude and originally pointing in the reference direction, so as to rotate it counter-clockwise through an angle θ; that is, $\cos\theta + j\sin\theta$ is a vector operator like (-1) and j, except that it results in a rotation through an angle of θ instead of 180° or 90°. We may then write

$$OA = (\cos\theta + j\sin\theta)OA'$$

In general if it is desired to operate upon a vector X so as to produce a rotation of θ, we indicate the operation by writing $(\cos\theta + j\sin\theta)X$. As a check let it be desired to rotate X through 90°. We may indicate this as $(\cos 90° + j\sin 90°)X$, which upon substitution of $\cos 90° = 0$ and $\sin 90° = 1$, reduces to jX, as it should.

The operator $(\cos\theta + j\sin\theta)$ may, however, be more conveniently written by making use of two trigonometric formulæ; namely,

$$2j\sin\theta = \epsilon^{j\theta} - \epsilon^{-j\theta} \tag{4}$$

$$2\cos\theta = \epsilon^{j\theta} + \epsilon^{-j\theta} \tag{5}$$

from which we obtain by addition,

$$\cos \theta + j \sin \theta = \epsilon^{j\theta} \tag{6}$$

If a vector of fixed length is to be rotated continuously and with constant angular velocity, it becomes necessary merely to make θ in the above operator an angle of constantly increasing size by writing $\theta = \omega t$. Let the fixed vector be of length E and let it start at zero time from the positive direction of the axis of reals. Then the rotating vector OA is

$$OA = (\cos \omega t + j \sin \omega t)E = E \cos \omega t + jE \sin \omega t.$$

The rotating vector is then seen to be composed of two vectors, one along the axis of reals and varying as a cosine, and the other along the axis of imaginaries and varying as a sine. But $OA = E\epsilon^{j\omega t}$, so that we may use $E\epsilon^{j\omega t}$ as a general expression, recognizing that its real part is a cosine function, of which the maximum value is E, and its imaginary part is a sine function of the same maximum amplitude.

Phase.—Consider the two expressions

$$E_1\epsilon^{j(\omega t+\theta)} = [\cos(\omega t + \theta) + j \sin (\omega t + \theta)]E_1$$

and

$$E_2\epsilon^{j\omega t} = [\cos \omega t + j \sin \omega t]E_2.$$

The first expression represents a rotating vector which is always ahead of the second vector by the angle θ. The angle θ is then said to represent a difference in phase between the two vectors. If θ is negative, the second vector leads the first, that is, the first mentioned vector lags behind the second.

In this case we have arbitrarily taken the second vector as a reference vector and assumed it to start its rotation at $t = 0$. The two vectors need not, however, be referred, one to the other, but both might be referred to some arbitrarily assumed vector which starts at a time $t = 0$, in which case both vectors would be represented by expressions of the same form as that of the first vector above, except that the phase angle θ would have different values in the two cases.

It is convenient to recognize that

$$\epsilon^{j\frac{\pi}{2}} = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2} = j \tag{7}$$

and

$$\epsilon^{-j\frac{\pi}{2}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j \tag{8}$$

and hence

$$\epsilon^{j\left(\omega t+\frac{\pi}{2}\right)} = \epsilon^{j\omega t}\epsilon^{j\frac{\pi}{2}} = j\epsilon^{j\omega t} \tag{9}$$

and

$$\epsilon^{j\left(\omega t-\frac{\pi}{2}\right)} = \epsilon^{j\omega t}\epsilon^{-j\frac{\pi}{2}} = -j\epsilon^{j\omega t} \tag{10}$$

Equations (9) and (10) thus indicate two ways in which a phase difference of 90° may be represented.

Conjugate Vectors.—The vector $v_1 = E'\epsilon^{j\omega t}$ has been shown to be a vector of length E' which revolves counter-clockwise with an angular velocity of ω radians per second. This vector is also to be considered as the sum of two vectors which do not rotate but which undergo sinusoidal variations in magnitude, one directed along the axis of reals and of value at any instant $E'\cos\omega t$, and the other directed along the axis of imaginaries and of value $E'\sin\omega t$. Thus

$$v_1 = E'\epsilon^{j\omega t} = E'\cos\omega t + jE'\sin\omega t \tag{11}$$

A vector of the same length, also starting from the axis of reals, and rotating with the same velocity but clockwise would be

$$v_2 = E'\epsilon^{-j\omega t} = E'\cos\omega t - jE'\sin\omega t \tag{12}$$

The sum of these two vectors is

$$v = v_1 + v_2 = E'\epsilon^{j\omega t} + E'\epsilon^{-j\omega t} = 2E'\cos\omega t \tag{13}$$

Equation (13) indicates that the sum of two rotating vectors which are equal in magnitude and velocity but opposite in direction of rotation is a vector which does not rotate. This vector is directed along the axis of reals and it varies in magnitude

sinusoidally, starting at $t = 0$ with its maximum value of 2 The two vectors and their sum at a time $t = t$ are shown Fig. 8.

These two rotating vectors which differ only in their direction of rotation are called conjugates. If either vector of a pair of conjugate vectors is given, then the other vector is obtained at once by reversing the algebraic sign of j wherever it occurs in the expression for the given vector.

Vector Representations of Alternating Currents and E.m.f.'s.—When an alternating e.m.f. is applied to an electrical conductor each carrier of electricity, *e.g.*, an electron, is urged first in one direction along the conductor and then in the opposite direction with a force which varies sinusoidally. If the maximum force acting upon the electrons is $2E'$ and if the frequency is $\omega/2\pi$ then the instantaneous value of the e.m.f. may be represented by v of equation (13) above. This follows at once from the fact developed above that v represents a sinusoidal variation along a fixed line.

Similarly, a sinusoidal alternating current of maximum value $2I$ may be represented by a pair of conjugate rotating vectors. Thus

$$i = I\epsilon^{j\omega t} + I\epsilon^{-j\omega t} = 2I \cos \omega t \tag{14}$$

This method of representing sinusoidal currents and e.m.f.'s avoids all the cumbersome trigonometric expressions and transformations which frequently obscure the physical relations involved. Thus the reader who is already familiar with the usual trigonometric methods may compare the two methods in Problem 14, page 191 and the footnote of page 28. The method involving vector operations will be seen to result in greater simplicity, also, wherever "rates of change" are concerned. Although a symbol "j," commonly known as an imaginary, has been introduced it will be remembered that the conjugate vectors of equations (13) and (14) were shown to represent real currents and real e.m.f's.

In equations (13) and (14) v and i have their maximum values at the moment when t is zero, that is, at the instant at which we begin to consider their variations. If it is desired to have the e.m.f. or the current start at $t = 0$ from a value of zero then it is necessary to write two new expressions, namely, v_1 and i_1 which shall differ from v and i of equations (13) and (14) in phase by 90°. Thus $E\epsilon^{j(\omega t - \pi/2)}$ is the expression for a vector 90° behind $E\epsilon^{j\omega t}$. But by equation (10) $E\epsilon^{j(\omega t - \pi/2)}$ may be written as $-jE\epsilon^{j\omega t}$. Similarly $E\epsilon^{-j(\omega t - \pi/2)}$ is a vector rotating clockwise which is 90° behind the clockwise vector $E\epsilon^{-j\omega t}$ that is $E\epsilon^{-j(\omega t - \pi/2)} = jE\epsilon^{-j\omega t}$. In the first case both vectors are rotating counter-clockwise as shown in Fig. 9. In the second case both vectors are rotating clockwise as shown in Fig. 10. Hence

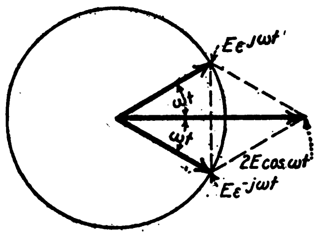
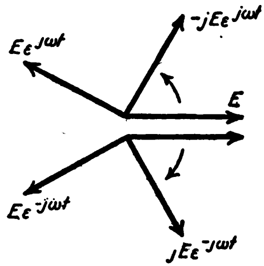


FIG. 8.—Conjugate vectors representing $2I$ cos ωt.



FIGS. 9 AND 10.—Conjugate vectors representing $2I$ sin ωt.

if

$$v_1 = E\epsilon^{j(\omega t - \pi/2)} + E\epsilon^{-j(\omega t - \pi/2)} \tag{15}$$

then

$$v_1 = -jE\epsilon^{j\omega t} + jE\epsilon^{-j\omega t} \tag{16}$$

or

$$v_1 = -jE(\cos \omega t + j \sin \omega t) + jE(\cos \omega t - j \sin \omega t)$$

$$= -j^2 \, 2E \sin \omega t = 2E \sin \omega t \tag{17}$$

Equations (15) and (16) are two equivalent expressions for the same pair of conjugate vectors.

In problems involving sustained alternating currents the

student is interested in a recurring cycle and it is immaterial at what point in the cycle he starts his consideration. In such problems it is therefore more convenient to consider time to start when the sinusoid is at its maximum value as in equations (13) and (14) rather than when it is at zero as in equations (15) and (16), since these expressions involve an extra use of the operator j. It is also evident that at times it may be necessary to deal with sinusoids starting at $t = 0$ from values other than either zero or a maximum. These cases will be discussed in a later section.

Rates of Change for Sinusoidal Functions.—It is shown in textbooks of mathematics that if

$$y = \epsilon^{at}$$

then

$$py = a\epsilon^{at} = ay.$$

This fundamental theorem may be accepted by the reader or he may verify it for himself after the manner of problems 15 and 16 of page 184.

In the expression $y = \epsilon^{at}$ it is important to remember that t, representing time, is what is called an independent variable, that y is the dependent variable and that a is a constant. The expression translated into words means that as t assumes different values, y also assumes different values which, however, depend upon those assumed by t. Corresponding to any value of t, say t_1, there is then a value of y, say y_1, which is found by raising the number ϵ (equal to 2.71828 +) to the power at_1 thus $y_1 = \epsilon^{at_1}$.

The rate of change of y is then the rate at which the dependent variable is changing with respect to the independent variable. The expression for this rate, namely $a\epsilon^{at_1}$ translated into words means that at just the instant when t has some value, say t_1, the quantity y, which at the moment is of value y_1, is changing or passing through this value at the rate $a\epsilon^{at_1}$ which is also expressible as ay_1. Similarly, when $t = t_2$, $y = y_2 = \epsilon^{at_2}$ and $py = a\epsilon^{at_2} = ay_2$.

Consider then an expression of the form $z = A\epsilon^{at}$ where

A is a constant, that is, is independent of t as to its value. It is evident that z is always A times as large as y. Hence also the rate of change of z, that is, the change in z per unit change in t, will be A times as large as the rate of change of y. Thus

$$pz = p(Ay) = Apy = A(a\epsilon^{at}) = aA\epsilon^{at} = az.$$

If for A and a we substitute E and $j\omega$ respectively and for z write v then for

$$v = E\epsilon^{j\omega t}$$

we have

$$pv = p(E\epsilon^{j\omega t}) = Ep\epsilon^{j\omega t} = E(j\omega\epsilon^{j\omega t}) = j\omega v.$$

Similarly if

$$v = E\epsilon^{-j\omega t} \text{ then } pv = -j\omega v.$$

Hence if

$$v = E\epsilon^{j\omega t} + E\epsilon^{-j\omega t} \text{ as in equation (13),}$$

then

$$pv = j\omega E\epsilon^{j\omega t} - j\omega E\epsilon^{-j\omega t} \tag{18}$$

also if

$$v = -jE\epsilon^{j\omega t} + jE\epsilon^{-j\omega t} \text{ as in equation (16),}$$

then

$$\begin{aligned} pv &= -jp(E\epsilon^{j\omega t}) + jp(E\epsilon^{-j\omega t}) \\ &= -j^2\omega E\epsilon^{j\omega t} - j^2\omega E\epsilon^{-j\omega t} \\ &= \omega E\epsilon^{j\omega t} + \omega E\epsilon^{-j\omega t} \end{aligned} \tag{19}$$

By comparison of the above with equations (9) and (10) it appears that if an alternating e.m.f. is represented by the sum of two conjugate vectors then the rate of change of this e.m.f. is represented by the sum of two conjugate vectors, each ω times as large as the vectors of the expression for the e.m.f. and each advanced 90° further in the direction in which it is rotating.

Inverse Rates of Change.—If $y = \epsilon^{at}$ then the expression $p^{-1}y$ means the quantity whose rate of change is y. Let this quantity be w, then $w = p^{-1}y$ or, $pw = y$ by definition of p^{-1}. It is evident that if w is put equal to y/a it will satisfy this condition since $pw = p(y/a) = \dfrac{py}{a} = \dfrac{ay}{a} = y$.

Hence for expressions of the form

$$
\begin{aligned}
i &= I\epsilon^{j\omega t} + I\epsilon^{-j\omega t} \\
p^{-1}\, i &= Ip^{-1}\epsilon^{j\omega t} + Ip^{-1}\epsilon^{-j\omega t} \\
&= \frac{I\epsilon^{j\omega t}}{j\omega} + \frac{I\epsilon^{-j\omega t}}{-j\omega} \\
&= \frac{-jI\epsilon^{j\omega t}}{\omega} + \frac{jI\epsilon^{-j\omega t}}{\omega} \qquad (20)
\end{aligned}
$$

Thus it is seen that the quantity whose rate of change is the sum of two conjugate vectors is itself the sum of two conjugate vectors, each $1/\omega$ part as large and each shifted 90° counter to its direction of rotation.

General Expression for Conjugate Vectors.—Consider the vector of the form $v = E\epsilon^{j(\omega t+\theta)} = (E\epsilon^{j\theta})\epsilon^{j\omega t}$ which was originally discussed on page 14. This vector is shown in Fig. 11. It is evident that it starts its rotation from a position θ radians from the reference line or axis of reals, since at $t = 0$ it

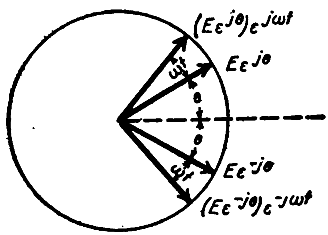


FIG. 11.—Conjugate vectors representing $2I \cos (\omega t + \theta)$.

equals $E\epsilon^{j\theta}$ as shown. The conjugate of this vector which is to have a negative rotation must then start as shown in Fig. 11 from the position $E\epsilon^{-j\theta}$. The sum of these two conjugate vectors is a sinusoidal variation along the axis of reals but it starts at $t = 0$ with the value $2E \cos \theta$ as is evident. Now consider

$$v = E\epsilon^{j(\omega t+\theta)} + E\epsilon^{-j(\omega t+\theta)} = E\epsilon^{j\theta}\epsilon^{j\omega t} + E\epsilon^{-j\theta}\epsilon^{-j\omega t}$$

Substitute for $\epsilon^{j\theta}$ and $\epsilon^{-j\theta}$, giving

$$v = [E \cos \theta + jE \sin \theta]\, \epsilon^{j\omega t} + [E \cos \theta - jE \sin \theta]\, \epsilon^{-j\omega t}$$

This is in the form of

$$v = E'\epsilon^{j\omega t} + \bar{E}'\epsilon^{-j\omega t} \tag{21}$$

$$= (E_1 + jE_2)\epsilon^{j\omega t} + (E_1 - jE_2)\epsilon^{-j\omega t}$$

where $E' = E_1 + jE_2 = E \cos \theta + jE \sin \theta$ (22)

and $\bar{E}' = E_1 - jE_2 = E \cos \theta - jE \sin \theta$ (23)

the symbol $\bar{E}'$ being used to represent the conjugate of E'. The general expression for a pair of conjugate vectors, as we see by reference to Fig. 11 and to the above expressions, comprehends a vector E' rotated counter-clockwise by the operator $\epsilon^{j\omega t}$ and a conjugate vector $\bar{E}'$ rotated clockwise by $\epsilon^{-j\omega t}$.

The sinusoid of equation (13) is obtained from equation (21) by placing θ equal to zero. The sinusoid of equation (17) is obtained by placing θ equal to minus $\pi/2$.

If for the E or for the I of the second vector of the pair in all the expressions for conjugate vectors that have occurred in the previous sections we substitute $\bar{E}$ and $\bar{I}$ respectively the expressions concerned become perfectly general and are no longer limited to the two special cases previously considered. Thus for equation (13) write

$$v = E\epsilon^{j\omega t} + \bar{E}\epsilon^{-j\omega t} = (E_1 + jE_2)\epsilon^{j\omega t} + (E_1 - jE_2)\epsilon^{-j\omega t} \tag{24}$$

then $pv = j\omega E\epsilon^{j\omega t} - j\omega\bar{E}\epsilon^{-j\omega t}$

$$= j\omega(E_1 + jE_2)\epsilon^{j\omega t} - j\omega(E_1 - jE_2)\epsilon^{-j\omega t}$$

$$= -\,\omega(E_2 - jE_1)\epsilon^{j\omega t} - \omega(E_2 + jE_1)\epsilon^{-j\omega t}$$

If $E_1 = E$ and $E_2 = 0$ as in the original of equation (13), then

$$pv = -\,\omega(-\,jE_1)\epsilon^{j\omega t} - \omega(+\,jE_1)\epsilon^{-j\omega t}$$

$= j\omega E\epsilon^{j\omega t} - j\omega E\epsilon^{-j\omega t}$ as in equation (18)

Omission of a Conjugate Vector.—Since as shown above the operation of finding the rate of change of two conjugate vectors

results in a new pair of conjugates and since the conjugate of a given vector is always to be obtained by reversing its rotation, that is by changing the algebraic sign of j wherever it occurs, it is evident that greater brevity of expression may be obtained by writing only one vector of the pair. Thus equation (24) may be written $v = E\epsilon^{j\omega t} = (E_1 + jE_2)\epsilon^{j\omega t}$ with the understanding that the conjugate has been omitted.

Now it will be remembered that the addition of the conjugate balances out the imaginary component of the first vector so that the sum of the conjugates represents a real sinusoidal quantity like a current or an e.m.f. If, therefore, one vector of the pair is omitted for convenience it must be done with the full understanding that such action is not rigorous, that the single term involves both a real and an imaginary components, and that to represent a real alternating quantity the conjugate must be introduced into the final result of any operations that may have been performed on the single vector.

For many problems it is sufficient to write the single vector, letting its real component represent the given current or e.m.f. and then in the final result to let the real component represent the desired current or e.m.f. This procedure is only safe, however, in solving problems of a type for which the rigorous solution, using the pair of conjugates, is known to lead to the same results as the method under discussion.

There is, moreover, *one type of problem where it is always necessary actually to deal with both vectors of the pair.* These problems involve either the product of two sinusoids or powers of the same sinusoid. It happens that in problems of this type the vector solution, even with the use of both terms, is in general not only shorter but free from all trigonometry and calculus and gives in simple vector form the desired results. For such an instance the reader is referred to page 61.

Vector Impedance.—In a preceding section the impedance of a circuit was defined as the ratio of the e.m.f. impressed upon it to the resulting current. When, however, storage reservoirs

for energy exist in the circuit the current is not directly proportional to the impressed e.m.f. In fact, when the e.m.f. is zero a current may still be flowing due to the redistribution of the energy of one or more reservoirs. At such an instant the ratio v/i is $0/i$ and the impedance is zero. Similarly, the current may be zero at an instant when the e.m.f. is not zero. At such an instant the ratio v/i becomes $v/0$ and the impedance is infinite. Thus consider Fig. 12 where are represented an alternating e.m.f. and a resulting current. At times corresponding to the points marked 0, ∞, $-\infty$, the impedances are zero, infinity, and minus infinity respectively. The instantaneous impedance varies then from $-\infty$ to $+\infty$.

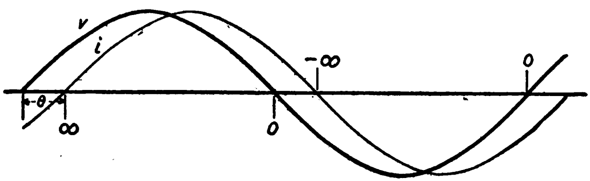


FIG. 12.—Variation of v/i with time.

It is evidently inconvenient to deal with the instantaneous impedance as defined above. What in most cases it is desired to know is first, the ratio of the maximum ampltude, E, of the impressed e.m.f. to the maximum amplitude, I, of the current, and second, the phase difference, θ, which the circuit occasions between the e.m.f. and the current. Assume for the moment that the e.m.f. leads the current. Then if the current is

$$i = I\epsilon^{j\omega t} + I\epsilon^{-j\omega t} = 2I \cos \omega t \tag{25}$$

the e.m.f. will be represented by a pair of conjugate vectors each E/I times as large as those representing the current and each *advanced in its direction of rotation* by θ. Thus the e.m.f. will be

$$v = \frac{E}{I} I\epsilon^{j(\omega t+\theta)} + \frac{E}{I} I\epsilon^{-j(\omega t+\theta)} = 2E \cos (\omega t + \theta)$$

or

or

$$v = Z'\epsilon^{j\theta} I\epsilon^{j\omega t} + Z'\epsilon^{-j\theta} I \epsilon^{-j\omega t} \tag{26}$$

where Z' is the numerical value of E/I.

The quantity $Z'\epsilon^{j\theta}$ is called the vector impedance, and will be represented by Z. The quantity $Z'\epsilon^{-j\theta}$ is evidently the conjugate of Z and is represented by $\bar{Z}$. We may also write

$$Z = Z'e^{j\theta} = \frac{E}{I} \cos \theta + j\frac{E}{I} \sin \theta \tag{27}$$

In this form it is usual to call $\frac{E \cos \theta}{I}$ the resistance and $\frac{E \sin \theta}{I}$ the reactance component of the vector impedance.

If, in arriving at the definition of Z as above, we had dealt with a current expressed in the more general form

$$i = (I_1 + jI_2)\epsilon^{j\omega t} + (I_1 - jI_2)\epsilon^{-j\omega t}$$

or

$$i = I\epsilon^{xt} + \bar{I}\epsilon^{\bar{x}t} \tag{28}$$

where

$$x = j\omega \text{ and } \bar{x} = -j\omega.$$

Then the e.m.f. would have been

$$v = ZI\epsilon^{xt} + \overline{ZI}\epsilon^{\bar{x}t} = E\epsilon^{xt} + \bar{E}\epsilon^{\bar{x}t} \tag{29}$$

where

$$Z = \frac{E_1 + jE_2}{I_1 + jI_2} = Z'\epsilon^{j\theta} \tag{30}$$

and

$$\bar{Z} = \frac{E_1 - jE_2}{I_1 - jI_2} = Z'\epsilon^{-j\theta} \tag{31}$$

where

$$Z' = \sqrt{\frac{E_1^2 + E_2^2}{I_1^2 + I_2^2}} \tag{32}$$

Equations (28) and (29) are in the most general form. They reduce to equations (13) and (14) for $E_1 = E$, $E_2 = 0$, $I_1 = I$, and $I_2 = 0$. Similarly for the same conditions, Z' of equation (32) reduces to Z' as given in connection with equation (26).

CHAPTER II

THE TELEPHONE RECEIVER

Magnetism.—With magnets in the form of compass needles and with the existence in and around the earth of a magnetic field, the direction of which at any point is indicated by the compass needle, the reader is, of course, familiar. He is also assumed to be familiar with the idea of north and south poles and with the fact that like poles repel and unlike attract. The direction of a magnetic field at any point is that in which a north-seeking pole would move. A magnetic field is then represented by lines of force which at any point show the direction of the field. The mechanical force which at any point would be exerted upon a unit of magnetism is the field strength. For convenience the lines of force are assumed to be drawn or to exist so that the number passing through any square centimeter of area (normal to their direction) is equal to the field strength at that point. The total number of lines crossing any area is the flux.

When a magnetizable body such as iron is placed in a magnetic field, the number of lines of force is increased. The ratio of the number per square centimeter of normal area, called the induction, to the number existing before the iron is introduced, that is, the field strength, is called the permeability of the iron. The permeability of a non-magnetic substance is unity. In that case the induction or flux per square centimeter is the same as the field intensity. For magnetizable substances the permeability is variable, depending upon the previous magnetic history of the substance and upon the induction in it.

If a gap is formed in a magnetic circuit, as for example, the

gap between a horseshoe magnet and its keeper, as shown in Fig. 13, the mechanical force exerted across the gap, that is, the pull of the magnet upon its armature, is found to be proportional to the square of the induction across the gap.

Magnetic Effect of a Current.—An electric current establishes in its vicinity a magnetic field. If the conductor carrying the current is long and straight, then the lines of magnetic force lie in planes perpendicular to the conductor and form concentric circles with it. The direction of the current (conventionally taken as opposite to the actual direction of motion of the elec-

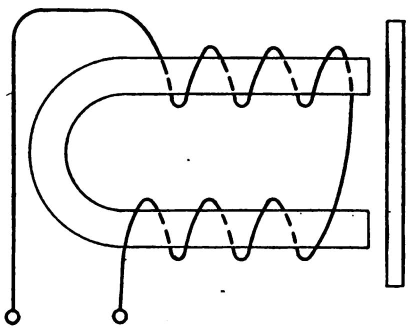


FIG. 13.—Telephone receiver.

trons) and the direction of the magnetic lines about it are related, as are the forward motion of a right-handed screw and the direction of rotation necessary to produce this forward motion.

If the conductor is bent into a loop, then the lines of force are no longer concentric with the conductor but are distorted by their interactions, tending to be closer together within the loop. In passing through the loop they all have the same direction. A single loop of wire carrying a current acts then like a flat disc magnet so far as attracting or repelling either a magnet or another loop carrying a current. This force in the case of two loops is proportional to the product of the two currents.

If the conductor is wound as a helix, that is, spirally, the result

is practically that of a succession of flat loops and hence the helix acts like a bar magnet. The direction of the magnetic field within the helix is found by assuming the helix to be grasped with the right hand in such a way that the fingers point in the direction in which the current is flowing in the wires, the direction of the magnetic field being then that of the thumb.

The Telephone Receiver.—The telephone receiver is essentially the horseshoe magnet and armature of Fig. 13, upon which is impressed a magnetic field due to the current in a helix wound around part or all of the horseshoe. If at any instant the induction across the gap is b, then the force pulling the armature or diaphragm toward the magnet is kb^2, where k is a constant.

In general the horseshoe is a permanent magnet and when no current is flowing through the receiver winding there is an induction of B. The total induction at any instant, namely b, is then the sum of B and the contribution to the induction made by the current. Neglecting changes in the permeability of the iron, this contribution will be proportional to the current in the receiver winding; that is, it will be $k_1 i$, where i is the instantaneous value of the current and k_1 is a constant. The pull on the diaphragm is then

$$kb^2 = k(B + k_1 i)^2 = kB^2 + 2kk_1Bi + kk_1^2i^2.$$

If no current flows, the pull is a steady one of value kB^2. If the horseshoe magnet is removed and a non-magnetizable horseshoe substituted, B is zero and the pull due to the current alone is $kk_1^2i^2$.

The total pull is then made up of three terms, one representing the pull due to the permanent magnetism alone, one representing the pull due to the current alone, and the product term, $2kk_1Bi$, representing the pull due to the superposition of two magnetic fields, namely, those due to the current and to the permanent magnet.

If it is desired that the diaphragm of the receiver shall have exerted upon it a force always directly proportional to the

current, so that its motions may reproduce for the human ear the variations in the received current, then the receiver must be so designed that this product term is important, and of the remaining terms, one, namely kB^2, constant, and the other negligible. This is obviously accomplished by making the induction due to the permanent magnet large, so that B is large as compared to k_1i.

When the receiver is so designed, the displacement of the diaphragm is proportional to k_1i, but is much greater than k_1i by a factor of B. The greater B is made (other things being equal) the greater becomes the displacement of the diaphragm for the same input current. In other words, we amplify the effect of k_1i by B. The receiver diaphragm then repeats the variations in the current but in an amplified form. This fact of repetition and amplification is of importance, as will be seen later.

If the square term of the current is not absolutely negligible as compared to the product term, then while there is repetition with amplification, there is also a distortion, in that the motion of the diaphragm is not directly proportional to the current at each instant but is greater than direct proportionality requires by the amount $kk_1^2i^2$.

It is of interest to consider the nature of this distortion for the special case where the current i in the receiver winding is of the form $I \sin \omega t$. The distorting pull is then $kk_1^2I^2 \sin^2 \omega t$, or $kk_1^2I^2(1 - \cos 2\omega t)/2$. It varies then from zero when $2\omega t$ is 0 or 360°, etc., to $kk_1^2I^2$ when $2\omega t = 180°$ or 540°, etc. This pull is then a pulsating one, equivalent to a steady pull of $kk_1^2I^2/2$ and a sinusoidal pull, namely $(kk_1^2I^2\cos 2\omega t)/2$, which is of twice the frequency of the input current.[1]

Actual Receiver.—Because of the fact that the diaphragm of a receiver is a stretched elastic body, it tends to vibrate most easily at a frequency, determined by its construction and elastic

[1] This is the trigonometric expression of the theorem developed on page 190 in analyzing the square of $I(\epsilon^{j\omega t} + \epsilon^{-j\omega t})$.

properties, which is known as its natural frequency. If the input current to a receiver is kept constant in amplitude but if its frequency is varied, it will be found that the greatest response or motion of the diaphragm occurs when the impressed frequency is the same as the natural frequency.

The natural frequency will of course depend upon the design of the mechanical system of the receiver. A typical case is shown in Fig. 14, of page 31.

Effective Resistance.—The telephone receiver is an especially good illustration of a circuit of variable impedance. To understand it better, it will be necessary to develop briefly the concepts of effective resistance and later of motional impedance.

Consider first the case of a receiver winding wound on a non-magnetic but conducting horseshoe, *e.g.*, a brass *U*. A steady unidirectional e.m.f. impressed upon the winding will cause a definite current and the ratio of this e.m.f. to its current is the direct-current impedance or ohmic resistance of the winding.

If, however, the impressed e.m.f. changes, the current changes and also the magnetic field around the winding. An e.m.f. of self-induction is then called into existence lasting as long as the current is changing in value (and of course opposing this change and thereby delaying the final steady state). The product of this e.m.f. of self-induction at any instant and the current measures the power expended in producing this change. When the current has become steady, it will be evident that the magnetic field about it has changed and hence that the energy stored in this field is greater or less than before. This change in energy is equal to the work done against the e.m.f. of self-induction. The alternating e.m.f. required to overcome the self-induction leads its current by 90° and hence the average power for this case is zero (as may be seen by considering the equation for power, developed in Problem 14 of page 191).

Neglecting, however, for the moment the phenomenon of self-induction let us note that the changing magnetic field induces e.m.f.'s in the brass horseshoe and these e.m.f.'s cause

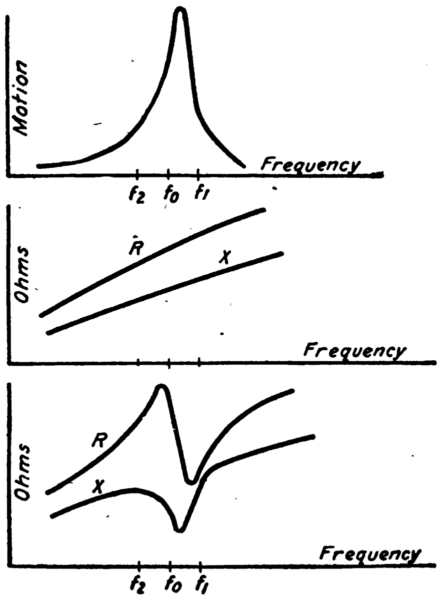
local currents, called eddy currents. In the resistance of the paths offered to these eddy currents there is then a dissipation of energy which must be supplied by the impressed e.m.f. These eddy currents of course induce counter e.m.f.'s in the receiver winding which must be overcome by the impressed e.m.f. exactly as in the case of the e.m.f. of self-induction and they represent a power expenditure in the same way, except that the power expended against the e.m.f. of self-induction leads to a storage of energy in the magnetic field, while in the case of the eddy current the energy is dissipated in heating effects. Such eddy-current effects accompany any change in current in the receiver winding and are therefore particularly noticeable in the case of an alternating current. Further, for the same maximum amplitude of alternating current in the winding, the eddy-current energy losses will increase with the square of the frequency since this energy is proportional to the square of the eddy e.m.f.'s and these in turn, being proportional to the rate of change of magnetic flux through the brass, are proportional to the frequency.

If now the brass horseshoe is replaced by the iron one it is evident the same phenomenon occurs. In addition, however, in the case of the iron there are losses due to hysteresis. When some of the molecular magnets of which the iron is composed are caused by the alternating magnetic field to turn, first in one direction and then in the other direction, work must be done. This work depends upon the magnetic condition of the iron at each instant, that is, upon the induction and upon the frequency with which these alternations occur.

The total work done by an alternating current flowing in the winding of a receiver (neglecting for the moment any motion of the diaphragm) is then seen to be made up of three parts, namely, one part due to the work done in overcoming the ohmic resistance of the winding, one part in establishing eddy currents, and one part in supplying the hysteresis losses in the iron. This work

all appears as heat losses in the various parts of the electrical and the magnetic circuits.

It is convenient to group all these losses of energy together and to define the effective resistance of a circuit as that resistance which an imaginary circuit free from eddy-current and iron losses must have in order that, for the same effective value of



FIGS. 14, 15 AND 16.—Characteristics of a telephone receiver.

the current, the heat losses in the two circuits shall be the same. The effective resistance of an alternating-current circuit is then the factor by which the square of the effective current must be multiplied in order to give the total power losses.

Because of the fact that eddy-current and iron losses increase with frequency, the effective resistance will increase with frequency much as shown by R in Fig. 15. This figure also shows

the numerical values of the reactance X, that is, the portion of the impedance due to the self-induction.

Motional Impedance.—So far in this discussion we have neglected the motion of the receiver diaphragm. The study of this motion may well be preceded by a consideration of some of the reactions in a direct-current motor.

To every action there is an equal and opposite reaction. In fact, we frequently measure the action by the reaction. Thus in exerting a force as by throwing a ball one is made conscious of the exertion by the reaction of the ball against his hand.

In the case of a direct-current motor the armature turns as a result of the force which is exerted by the action upon each other of the two magnetic fields, namely, that due to the current in the armature and that due to the poles of the magnet. The current and hence the action is due to the impressed e.m.f. The reaction appears as a counter e.m.f. opposing the flow of current through the armature and, as is well known, making it less than it would be with the armature at rest, as for example, at the time of starting. This counter e.m.f. is of course induced in the armature as a result of its motion in the magnetic field of the pole pieces. The ratio of this counter e.m.f. to the current is an impedance which is called into effect by the motion of the armature and might therefore be called a motional impedance.

In any direct-current machine, whether running as a generator or as a motor, there is induced in the armature an e.m.f. of $E_a = KS$ where K is a constant and S is the speed, *e.g.*, in r.p.m. If a current of I is flowing through the armature the power is E_aI. If this power is positive, the machine is acting as a generator, if negative as a motor, in which case the direction of flow of current is against the induced e.m.f. or, in other words, the induced e.m.f. is then a counter e.m.f. Note that for a given value of current the power is greater the greater the speed, and that for a given speed the power is greater the larger the current.

If an e.m.f. of E is impressed upon the armature whose resistance is R then $E - E_a$ that is, $E-KS$, is the resultant e.m.f.

which forces a current of I through R. That is

$$E - KS = RI \text{ or } E = RI + IKS/I$$

Hence

$$E/I = R + KS/I$$

From this expression we see that the impedance of a direct-current motor may be written as the sum of two impedances, one the resistance R and the other the motional impedance KS/I. This motional impedance is not a constant but depends upon the motion, being greater for the case of greater response, that is, speed (other things being equal). This motional impedance is of course measured in ohms and in the case just considered is in phase with the resistance.

In the case of the telephone receiver the motion of the diaphragm varies the reluctance of the magnetic circuit and so alters the flux through the receiver winding, thus inducing in it an e.m.f. If one talks into a receiver the output current from the receiver winding reproduces the variations of the voice and the receiver is similar in action to a generator. If the motion of the diaphragm takes place as the result of a current in the receiver winding, the induced e.m.f. is a counter e.m.f. and the case is one of motor action.

To determine the motional impedance of a direct-current motor it is necessary merely to measure E and I and subtract R from E/I, giving KS/I as above. The resistance R should be found experimentally by blocking the motor so that it will not turn and dividing the impressed e.m.f. by the armature current.

Similarly for a telephone receiver except that in this case both impedances are complex instead of being pure resistances. Fig. 15 shows a typical relation for the components of the impedance of a blocked telephone receiver. Fig. 16 shows the total impedance when the block is removed and motion is allowed. The difference between the ordinate of the two curves of R (Figs. 15 and 16) for any frequency gives for that frequency the real component of the motional impedance. And similarly the

difference between the two curves of X gives its imaginary component.

Let M_1 and N_1 be the real and imaginary components of the motional impedance at a frequency f_1. Then the vector impedance for that frequency may be plotted as in Fig. 17. If this is done for several frequencies it appears that the locus of the extremities of these vectors is a circle, as in Fig. 18.

Consider now the power expended in the receiver. Let $R_1 + jX_1$ represent at frequency f_1 the impedance of the blocked

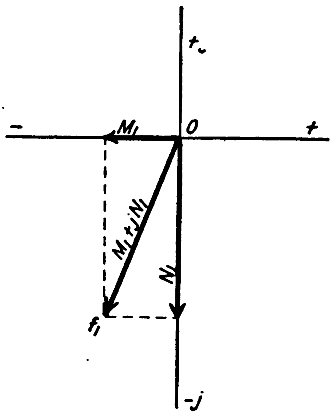


FIG. 17.—Motional impedance of a telephone receiver.

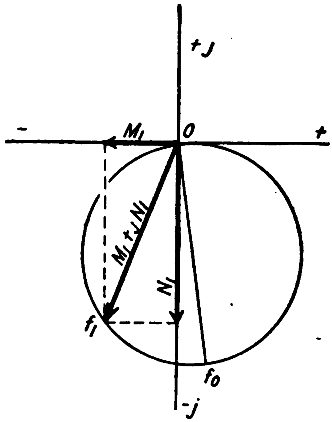


FIG. 18.—Circle diagram for motional impedance.

receiver. The total impedance is then $(R_1 + M_1) + j(X_1 + N_1)$. The e.m.f. required to force a current I through the winding is I times this impedance. The power expended is this e.m.f. times the current. The average power expended is $EI \cos \theta$ as in Problem (14) of page 191. However, as may be shown, the power, $EI \cos \theta$, is I^2 times the real component of the impedance, hence is $(R_1 + M_1)I^2$ where I is the effective value of the alternating-current input. Writing this product as $R_1I^2 + M_1I^2$ it is seen that the power is composed of two parts, one, the losses in the effective resistance of the winding and the other, the power

expended in driving the receiver diaphragm. Below the natural frequency, M_1 is positive and we have motor action. Above this frequency, M_1 is negative and power is returned to the receiver winding by the moving diaphragm; that is, we have generator action.

This is somewhat analogous to the case of an induction motor which runs, as is well known, as a motor below synchronism with the impressed e.m.f. but acts as a generator when driven above synchronism, delivering energy to the input line.

For most practical purposes we are concerned in wireless work only with the fact that for any receiver with an elastic diaphragm there is a natural frequency at which the response for a given current input is greatest. It is well to bear in mind, however, that in the neighborhood of this frequency the impedance of the receiver varies greatly as the frequency is changed. In the case of the receiver the impedance is composed of two impedances, one, that of the receiver winding and the other, that of the reaction of the moving diaphragm. In the study of so-called coupled circuits we shall later meet a somewhat similar case in the fact that the impedance of a circuit may be considered to be its own impedance plus an impedance representing a reaction due to coupling with it a second circuit.

CHAPTER III

THE VACUUM TUBE

Conduction of Electricity.—The conduction of electricity through a medium separating two electrodes which are maintained at different electrical potentials by an e.m.f. is the result of the movement in the medium of "carriers" of electricity. The nature of these carriers is determined in part by the medium. Four cases may be distinguished, namely, solid media, liquid media, aeriform media (*i.e.*, gases and vapors) and vacua. Of these four, only the last two will be considered in this chapter.

Carriers of electricity may be classified first as to size and second as to the sign, positive or negative, of the charge which they carry. The smallest known carrier of electricity, called the "electron," has a definite amount of negative electricity which is not to be dissociated from it. In fact, this negative charge itself is to be considered as constituting the electron. In the structure of matter, these electrons are associated with nuclei of atomic size and form atoms which show no properties of electrical charges. The norm in matter is, then, the uncharged atom or molecular aggregate of atoms. A normal atom or molecule may, however, have dissociated from it by various means one or more of its component electrons. When this is accomplished "ionization" is said to have taken place. When an atom has lost in this way an electron it has, as viewed from the standpoint of the norm, a deficiency of negative electricity and therefore has the characteristic of a positive charge. In this condition it behaves as a carrier of positive electricity and is called an "ion." The dissociated electron is also a carrier of electricity and as such is sometimes spoken of as an "ion."

Preferably, however, it is to be called an electron and the term "ion" applied only to carriers of atomic or molecular size. Negative ions may be formed by the combination of a normal atom or group of atoms and an electron, but these are not of frequent occurrence in the phenomena under discussion and need not concern us further.

That a vacuum must be free of ponderable matter such as normal atoms and ions follows at once from definition. Whether or not it is justifiable to admit the presence in a vacuum of electrons is perhaps a disputable question, but considering that the electron can be made to pass through ordinary matter as in the case of cathode rays, and also considering that it is electricity rather than matter as we usually understand "matter," the definition of a "vacuum" for the purpose of this discussion will be "a space free from dislodged molecular or atomic masses." Of the two types of carriers defined above, it follows that only electrons can serve in conduction through vacua, while both electrons and ions may serve in conduction through aeriform media.

So far as concerns the mere passage of an electrical current through a circuit containing an aeriform medium, it is not essential under which of the following conditions or under what combination thereof conduction takes place in the aeriform medium. The possible conditions (neglecting the action of negative ions) are: (*a*) Conduction by the motion of electrons from the negative electrode (*i.e.*, cathode) to the positive electrode (*i.e.*, anode); (*b*) conduction by the motion of positive ions from the anode to the cathode; (*c*) conduction by withdrawing from the medium to the anode, electrons, and to the cathode, positive ions. The phenomena of light, heat and current intensity occurring within the aeriform medium are dependent upon the above conditions and also upon the means whereby these carriers are brought into existence.

From the preceding discussion it appears that an aeriform medium containing only normal atoms would be non-conducting

except as rendered so by the action upon it of some ionizing agent, or except as electrons might be supplied at the cathode or ions at the anode for conduction, after the manner described above. We have then three methods, by any of which or by a combination of which a non-conducting aeriform medium may be made conducting. They are: (1) The ionization of the medium; (2) the emission of electrons at the cathode; (3) the emission of positive ions at the anode.

It will be recognized that except in so far as free electrons are available for emission at the cathode, both ions and electrons are obtained by disrupting the normal atom. In disrupting atoms, equal numbers of ions and electrons are formed.

Of the several methods by which the atom may be disrupted, there may be mentioned: (*a*) Ionization by ultra-violet light; (*b*) ionization by Roentgen rays; (*c*) ionization by radioactive substances; (*d*) ionization by chemical action; (*e*) ionization by collision with ions or electrons; (*f*) ionization by incandescent substances; (*g*) spontaneous ionization.

It is not necessary to describe these methods in detail, since the basic principle of all except (*g*) seems to be that ionization occurs as the result of a stimulus or disturbance of the atom by contributions of energy derived from heat, electromagnetic radiation, or impacts with carriers. Even in a gas not subjected to any of the ionizing means named above, a small number of free ions and electrons is always present and is assignable in origin to a spontaneous disintegration of the atom.[1]

In the case of metals it has been held that there is always present a large number of free electrons. These electrons are, of course, available for conduction in the metal. However, they may not be separated from the mass of the metal, except by heat as in the case of solids heated to incandescence, or by the application of some other means, as for example exposure to ultra-violet light.

Upon the basis of the theory stated above, certain cases will

[1] The cause is probably penetrating radiations from the earth.

now be considered, namely: (1) Conduction through vacua; and (2) conduction through gases or vapors.

Conduction through a vacuum has been mentioned incidentally above. As stated, it can occur only as the result of the passage of electrons, since the passage of ions of atomic mass would violate the condition that the medium be a vacuum. The current then can pass only by the movement of electrons from the cathode to the anode. The only source of electrons, therefore, which avails for this conduction is the material of the cathode. This may be made to emit electrons by heating to incandescence or by illumination with ultra-violet light.

In conduction through gases, two cases should first be distinguished with reference to the condition of the medium at the instant that the conduction starts, namely: (*a*) when no ionizing agent is active; and (*b*) when such an agent is present. In case (*a*) conduction starts either (1) as the result of such carriers as may be present in the gas, whether these are due to the spontaneous disintegration of some atoms or are residual from some previous ionization, or (2) as the result of some mechanical means for temporarily short-circuiting the nonconducting path of the gas. In both cases if conduction is to be maintained, the initial passage of electricity must introduce a self-perpetuating ionization means. The statements of the two preceding sentences may be seen more clearly by considering illustrations.

As an illustration of case (*a*1) above, we have the spark discharge which occurs in air between electrodes of sufficient potential difference. The explanation of the spark is as follows: When the potential is applied to the electrodes the free electrons which are in the gas move immediately to the anode, and similarly the positive ions move to the cathode. The current corresponding to this movement is very small and is merely the usual leakage current of an otherwise efficient insulator. The greater the potential difference the greater the velocity with which this movement of carriers takes place. If it is great enough, some

of the moving ions may collide sufficiently hard with the molecules of the gas, particularly the layer adjacent to the cathode, to produce ionization. Of the electrons so formed many travel over the entire distance between electrodes and by impact with the intervening molecules, produce further ionization. In its initial stages, then, the discharge takes place as a spark through the ionized medium. If the potential difference is maintained, the spark is quickly followed by a sustained arc. Both the initial spark and the subsequent arc assist the continued ionization of the medium by the heat they generate.

As an illustration of case (*a*2), where mechanical means of starting are employed, we have the familiar case of striking an arc between two carbons by momentarily allowing them to touch and then slowly withdrawing them. In this case the intense heat liberated at the point of contact is sufficient to start ionization. Ionization, once having started, is maintained largely by the ionizing effect of the impact of the positive ions upon the surface of the hot cathode.

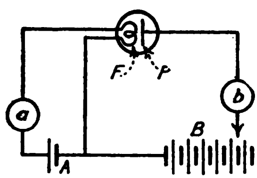


FIG. 19.—Two-electrode vacuum tube.

In case (*b*) where the ionizing agent is already active, conduction starts as soon as the potential is applied to the electrodes, and the passage of electricity is controlled largely by the efficiency of the ionizing agent and by the electrostatic field maintained between the electrodes. As illustrations of this type may be mentioned conduction through flames or through gases arising from flames, conduction through gases exposed to radioactive substances and also conduction in gaseous amplifiers of the Von Lieben type. In this last case the cathode is formed by an incandescent filament and is a constantly available source of electrons. These electrons serve both as carriers of the current between the electrodes and as ionizing agents acting upon the gas in the tube to produce other carriers.

The DeForest audion[1] and more especially the Western Electric Company's vacuum tube differ from this latter type in that the residual gas in the tube is so small in amount that the ionizing effect of the electrons emitted by the incandescent electrode is essentially negligible. In a device of this type the operation within the normal working limits is essentially that of conduction through vacua, since it takes place entirely through the passage of electrons liberated at the cathode.

Having outlined the modern theory for the conduction of electricity through aeriform bodies and vacua we will now discuss the characteristics of the three-element vacuum tube.

The Vacuum Tube.—Consider an evacuated vessel containing a filament F, which may be heated by current from a battery A as in Fig. 19. Let P be a plate which, by an external source B of e.m.f. of E_B volts, may be maintained at a potential different from that of F. Let an ammeter b indicate the current I_B flowing in the circuit from P to F inside the tube, and through B and b outside the tube. An ammeter, a, in the filament circuit indicates the current heating the filament. Now for a fixed value of the current I_A let a series of readings be made of I_B for various values of E_B. If these are plotted as in Fig. 20, the points will lie on a curve of the form odf given by the full line.

The current flows as a result of the attraction of the plate P for the electrons (emitted by the heated filament) and of the repulsion exerted by the filament F upon the electrons, that is, the electrons move from F to P as the result of an electrostatic field maintained between F and P by the e.m.f., E_B. This field is conveniently measured in terms of the voltage E_B, and when the resistance of the ammeter b and the battery B is small, it may be taken equal to E_B. As long as E_B does not exceed the value oc shown in Fig. 20, the number of electrons drawn from F to P per second, that is, conventionally the current from

[1] cf. DeForest, Trans. A. I. E. E. vol. 25, pp. 735–763, 1906, also DeForest, Proc. I. R. E. vol. 2, pp. 15–29, 1914.

P to F, is found by analysis of the curve *od* to be roughly proportional to the square of the field intensity. This is a convenient approximation of which much use will be made later.

For the moment, however, consider the effect when E_B is increased beyond *oc*. Remembering that current is the number of electrons transferred per second through a cross-section of the conducting path, it appears that if the field *oc* due to E_B is sufficient to transfer all the electrons emitted by the filament, a further increase in E_B cannot increase the current.

Such an increase in current can come only by increasing the number of electrons emitted by the filament, and hence available for producing a current between F and P, or an increase in available carriers may arise from ionization of any residual gas in the tube if E_B is, sufficiently increased. In a pure vacuum device the number of available electrons may be increased by increasing the heating of the filament. If the filament temperature is raised by increasing I_A to some new fixed value, it is found that for further increases of E_B the current is given by the dotted portion, *deg*, of Fig. 20. If then it is desired to utilize fields up to a strength represented by some definite value of E_B and to have the current follow the approximate square law mentioned above it becomes necessary to heat the filament by some definite value of I_A.

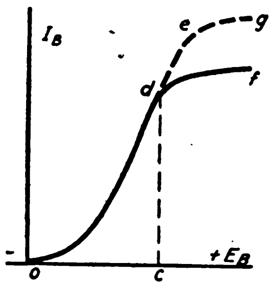


FIG. 20.—$E_B - I_B$ characteristic.

To find the required value of I_A, let E_B be given some definite value, and let a plot be made as in Fig. 21 of I_B for various values of I_A. The relation will then be found to be that of the curve *ohj* of this figure. After I_A has reached a value of *on*, there is no increase in the current I_B for further increases in the temperature of the filament.

To understand the phenomenon involved, consider the simple case of a metallic conductor *mn* and an electrostatic field repre-

sented in direction by the arrows of Fig. 22. When the conductor *m* is placed in the field, its electrons are urged from *n* toward *m*, and some of them do move in that direction. In other words, a momentary charging current flows in the conductor. This transfer of electrons continues until there is a sufficient number accumulated at *m* so that their repulsion upon any other electron, as one at *s*, is just equal to the force exerted on it by the original field. This then is a stable condition, in which the field due to the charges induced in conductor *mn* just equals at every point in this conductor the impressed field.

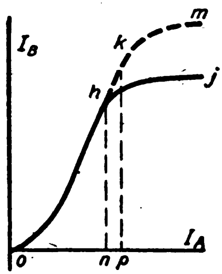


FIG. 21.—$I_B - I_A$ characteristic.

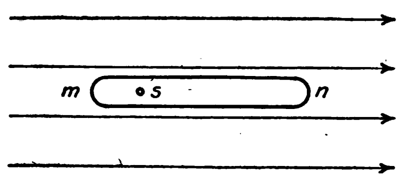


FIG. 22.—To explain charging of a conductor in a field.

If the external field is increased then equilibrium is destroyed and more electrons will be transferred until a new balance exists between the externally impressed field and the field due to the accumulated charges. Suppose, however, that the supply of electrons was limited; then there would be brought about a condition where the field at *s*, due to the accumulated electrons, did not balance the impressed field, and an increase in the impressed field could cause no charging current, because there would be no available carriers. If, under these conditions, an increase is made in the supply, as for example by some ionizing means, then a new charging current would flow until a new condition of stability were attained.

Now this hypothetical condition for a metallic conductor is essentially the condition within the vacuum tube, except for the

fact that an external conducting path is offered, so that electrons may travel from m to n. The electrons available in the space between filament and plate are supplied by a source of limited ability (in this case the filament). The ability of the source may be increased by increasing its temperature. For a definite value of the field E_B the number of available electrons increases with the filament temperature, as shown by the part oh of the curve of Fig. 21. When the temperature corresponding to $I_A = on$ is reached, the accumulation of electrons in the space between F and P is sufficient to neutralize the impressed field at F and to prevent a further increase in the number of electrons in the tube. Another condition of stability is therefore attained, under which no increase in current is to be obtained, even though the source of electrons is capable of supplying a larger number. Although there is no increase in the number of electrons and hence in the current, there is a constant current flow through the tube and the external conducting path. As fast as electrons are subtracted from the plate they are returned by the battery B to the filament and there emitted so as to maintain the condition of equilibrium.

When the source of electrons is capable of supplying a larger number of electrons, as for example when I_A in Fig. 21 is greater than on, then a greater current may be obtained by increasing the field. The partially dotted curve $ohkm$ of the figure shows the relations between I_B and I_A when the voltage E_B has been increased. If the filament is heated by a current less than on, it is evident that the plate-circuit current I_B does not increase with increased E_B. In order then that the vacuum-tube system shown in Fig. 19 should obey the square law, namely, I_B proportional to the square of the field intensity, it is necessary that the current I_A should be made as large as op, where the curve $ohkm$ is taken for the maximum value of E_B which it is expected to use with the tube.

Three-element Tube.—From the previous description of a simple vacuum-tube system involving two electrodes or elements,

namely, a filament source of electrons and a plate, it appears that I_B is proportional to the square of the field between the two electrodes. For the case just considered the field is entirely due to the battery B, and $I_B = k(E_B)^2$, where k is a constant. The field may, however, receive contributions from another source of e.m.f. if a third electrode is inserted as shown at G of Fig. 23. The relation between I_B and the field then becomes

$$I_B = k\,(E_B + k_1 v_1)^2, \tag{33}$$

where $k_1 v_1$ represents the contribution to the field of the third electrode, G. This grid is maintained at a potential different from that of F by a third source of e.m.f., namely, v_1.

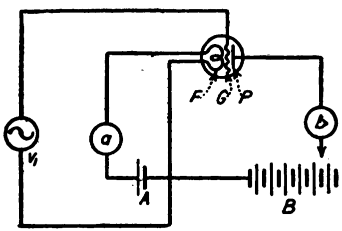


FIG. 23.—Three-element vacuum tube.

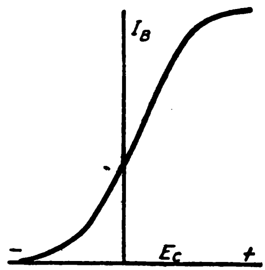


FIG. 24.—$I_B - E_C$ characteristic.

The equation[1] just written should be compared with that found in Chapter II for the telephone receiver, remembering, of course, that k and k_1 are merely general symbols for constants depending upon the design, thus:

$$\text{pull} = k(B + k_1 i)^2$$
$$I_B = k(E_B + k_1 v_1)^2.$$

The value of k_1 in the vacuum-tube equation is determined by the design of the tube.

Consider first the case when v_1 is due to a battery of E_C volts. The relation is then $I_B = k(E_B + k_1 E_C)^2$, which is a curve of the same form as Fig. 20. It is usually more convenient, however,

[1] This equation is obtained from some as yet unpublished researches of DR. H. J. VAN DER BIJL of the Western Electric Company.

to draw the curve showing the relation of I_B and E_C as shown in Fig. 24. This relation may be expressed as $I_B = a(gE_B + v_1)^2$ or $a(gE_B + E_C)^2$ when v_1 is E_C. The constants a and g are now used and obviously $g = 1/k_1$ and $a = k_1^2k$. When E_C is zero, then I_B is $ag^2E_B^2$. Also when E_C is $-gE_B$, then I_B is zero. The constant g then represents the fraction of the voltage E_B which must be applied to the grid, making it negative with reference to the filament, in order that the current in the plate circuit shall be reduced to zero. The reciprocal of g, that is, k_1, apparently represents the relative importance, in producing an effect upon the current I_B, of a voltage applied at G and one applied at P. In some commercial forms of the three-element tube k_1 will be found to have values lying between 5 and 40.

It is evident then that a small voltage at G is equivalent to a larger voltage at P. Hence this tube system may be used as an amplifier or means whereby a small cause may produce a large effect. The energy required is, of course, supplied by the A and B batteries, and there is no violation of the principle of the conservation of energy.

In general the voltage v_1 impressed on the input terminals of the tube, that is, on F and G, is made up of a constant voltage E_C and a variable voltage, say v. The current in the output circuit, that is, from F to P, is then expressible as

$$I_B = a(E_0 + v)^2 \tag{34}$$

where

$$E_0 = gE_B + E_C. \tag{35}$$

If $v = E \sin \omega t$, then just as for the telephone receiver the output (in this case I_B) will consist of four terms: thus

$$I_B = aE_0^2 + aE^2/2 + 2aE_0E \sin \omega t + E^2 (\cos 2\,\omega t)/2,$$

of which one represents an amplified repetition of the input and another is a double-frequency term. The vacuum tube may therefore be used to indicate, by a change in the value of the direct current flowing in the plate circuit, the fact that a

sinusoidal voltage is impressed on the grid, or to give a repetition of the input, or to produce a change in frequency.

Before considering further the above equation, we shall show qualitatively by reference to Fig. 25 the effect, for a definite sinusoidal input, of various values of E_C, upon the output. Thus if E_C is of value Oa, it is evident that when v is zero the current

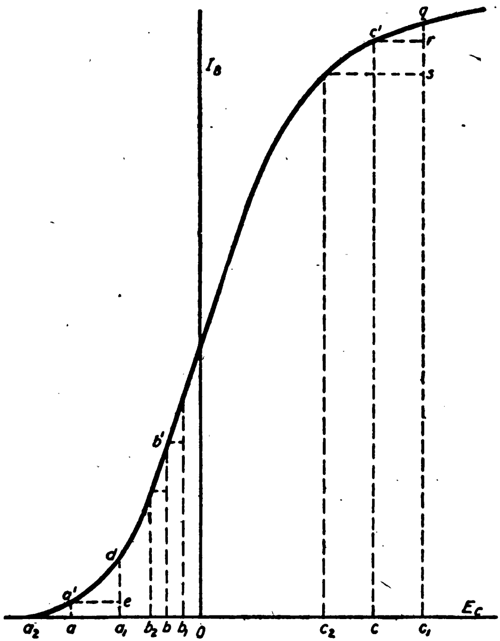


FIG. 25.—Dependence of output on E_C.

I_B is of value aa'. When v becomes E sin 90° $= E$ then I_B has increased to a_1d. When v becomes E sin 270° $= -E$, then I_B becomes zero. From the figure it appears that de is greater than ea_1, hence the output current is increased by the impressed sine wave of voltage more than it is decreased. The output is evidently a distorted sinusoidal current and its average value is greater than ea_1, which is the value of I_B before the voltage v is

impressed. Conversely if E_C is equal to Oc, the positive half of the input wave produces a change in current of rq which is smaller than the change rs produced by the negative half wave of v. The average current in the plate circuit is then less than cc'. Some value of E_C as Ob may, however, be found such that if E is not too large, the two half waves of alternating current will be symmetrical. For this condition there would be in the output circuit a repetition, free from distortion, of the impressed wave form. The average value of the current I_B would be bb', that is, the same as if v were not impressed.

Vacuum-tube Amplifier.—From the foregoing study of Fig. 25 it appears that the three-element vacuum tube may be used as a distortionless repeater, provided E_C is properly chosen. A criterion and test is the constancy of the direct current through the plate or output circuit, independent of the alternating voltage input.

The characteristics plotted in Figs. 20, 21, 24 and 25 are all upon the assumption that the external plate circuit is of negligible resistance, and hence that the field between P and F is independent of the plate current I_B. If, however, there is in the plate circuit a resistance R, the field is not proportional to E_B but instead to $E_B - I_BR$. If E_B is kept constant the equation for I_B becomes

$$I_B = a[g(E_B - I_BR) + E_C + v]^2$$

or

$$I_B = a(E_0 - gI_BR + v)^2. \tag{36}$$

Qualitatively it therefore appears that the curve of current given in Fig. 24 and reproduced in the full line of Fig. 26 would be more like the dotted curve if the resistance were made large. One method of reducing the effect of the double-frequency term, that is, one method of obtaining a linear relation or direct proportionality between the output and the input is then to insert a high resistance in the plate circuit.

For greatest output this resistance should be made equal to

the average internal resistance of the tube, just as in the case of a direct-current generator the greatest power output occurs when the external and armature resistances are equal.

The internal impedance of the tube between the output terminals P and F is essentially a pure or ohmic resistance. This

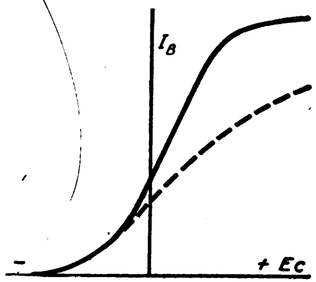


FIG. 26.—$I_B - E_C$ characteristic with resistance in the plate circuit.

resistance, however, as is evident from Figs. 20, 24 and 26, depends upon the field between P and F. In fact, it may be said that it is by virtue of this dependence of resistance upon the field that the device has the characteristics shown in the figures. The direct-current resistance of this output circuit is found by

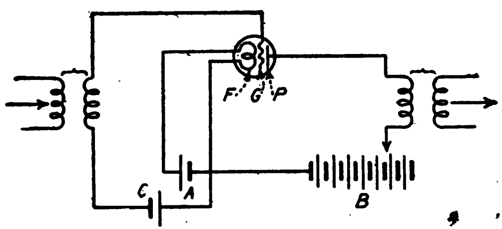


FIG. 27.—Typical vacuum-tube circuit.

dividing E_B by I_B. The internal impedance of the output circuit of the tube is to be found by taking the slope of the $E_B - I_B$ characteristic. This, of course, varies from point to point.

4

The input impedance, that is, the impedance between G and F, is infinite as long as G is negative with respect to F. This is evident when it is remembered that electrons can be drawn to G from F only in case G is positive with respect to F. When, however, G is made positive, then a current can flow. As long as $E_C + v$ is negative, however, the input circuit has an infinite resistance and absorbs no energy from the source of e.m.f. v.

While many different circuit arrangements are possible, it is necessary merely to understand one circuit in order to devise others. Thus Fig. 27 shows a possible scheme of connections for a vacuum tube. Whether this tube is operating as a detector or as an amplifier obviously depends upon the adjustments as described above. Transformers are shown for connecting the tube to the source from which v is to be derived, and to the apparatus where the amplified (or detected) current is to be utilized.

CHAPTER IV

DETECTION OF HIGH-FREQUENCY CURRENTS

Detection and Measurement of Current.—Consider an electrical circuit for which it is to be determined whether or not a current is flowing through it. To detect the existence of a current means to make it produce some effect on the senses of the observer. Thus the existence of a current is detected if the current is caused to produce an audible sound, a visible motion, a burning, pricking or tingling sensation, if it produces a sensation of taste, or if it affects the senses of smell. In the detection of currents the senses usually employed are of course those of sight and hearing.

To measure the current means of course to compare it with some standard current or unit and to express it as a certain number of times the unit. For purposes of measurement only the senses of sight and of hearing are sufficiently accurate to admit of the necessary comparisons between the unknown and a known current.

Using the sense of hearing, the method of measurement is practically limited to alternating currents the frequencies of which are well within the audible limits, say between 200 and 3000 cycles per second for the ordinary observer. Within such limits, however, the human ear is remarkably sensitive and very accurate comparisons may be made between two similar sounds to determine which of the two is the louder. It is thus possible to determine when two similar sounds are equal by listening to them alternately for a few moments, but the ear does not admit of a direct measurement in the sense of determining without intermediate apparatus how much greater one sound and hence one current is than another.

To measure an alternating current by the ear requires, then, that another alternating current of the same frequency and of known or assumed amount is available and also that means be provided for reducing one or the other of these currents by a determinable amount. Thus let it be desired to find the relative efficiency of two wireless detectors represented by A and B of Fig. 28. Arrange resistances in the outputs so that a part may be used as a shunt to the receiver. Arrange the circuit with two double-pole double-throw switches as shown. Listen first on one circuit and then on the other, adjusting the variable resistance until equal volumes of sound occur. The currents through

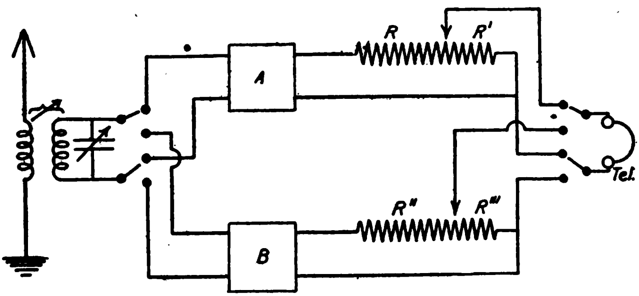


FIG. 28.—Comparison of detectors A and B.

the receiver are then the same and hence a ratio for the currents through R and R'' may be obtained by calculation. If R and R'' are much greater than R' and R''' respectively and if the impedance of the receiver is large as compared to either R' or R''' the ratio of the current output of A to that of B is inversely as R' is to R'''. Of course in any test of this sort the total resistance for each circuit should be that external resistance into which the circuit works most efficiently.

Visual indications of a current flow are not restricted when use is made of them for measurements as are audible indications. Different devices are required, however, depending upon the character, *i.e.*, wave form of the current and upon the sen-

sitiveness required. Current-measuring instruments with which the reader should be familiar from earlier reading are the moving-coil galvanometer, which in its commercial form is illustrated by the direct-current instruments of the Weston Company, the dynanometer which is illustrated in a commercial form by alternating-current instruments having one fixed and one movable coil, the plunger-type ammeter for alternating-current and the hot-wire ammeter.

Consider for the moment the moving-coil galvanometer. It consists of a coil or loop of wire of one or more turns placed in a permanent magnetic field. Its rotation, due to the reaction with this field of the current flowing in it, is proportional to the current. The rotation is manifested by the motion of a pointer attached to the coil, in an ammeter, or by the motion of a spot of light reflected from a mirror attached to the coil in the case of a galvanometer. In the latter case if the spot of light falls on a continuously moving photographic film it traces on a time axis the wave form of the current. The ordinates of this trace are proportional to the current and the abscissæ to the speed of the film. In this case the instrument is called an oscillograph and the photograph is an oscillogram.

In all these cases the coil (when the current is applied) turns from its zero position against a restoring force supplied by an elastic support or spring. If the natural frequency of the mechanical vibrating system thus formed is high and the damping large the coil motion will follow exactly the changes of a lower frequency current through the coil. If, on the other hand, the natural frequency is low, that is, if the time required for one full swing of the coil system is large it can not in its motion follow a rapidly alternating current. No sooner does it start to rotate in one direction than an alternation in the current imparts a tendency in the opposite direction. If a sine wave of current is impressed upon such an instrument the latter indicates zero current, that is, the average value of the sinusoidal current. If, however, a pulsating current is passed through the

instrument it will indicate the average value of the current, that is, the value of the direct-current component of the pulsating current. A current-measuring instrument of the moving-coil galvanometer type, therefore, indicates *average* values of the current in it except in those cases where its moving part has a high natural frequency and thus follows more or less directly the variations of the current, as in the case of the oscillograph.

The hot-wire instrument, on the other hand, consists merely of a length of conductor through which the current flows. This length alters due to the changes in temperature occasioned by the heating effect of the current. The changes of length are indicated by a pointer attached to the conductor through a lever system. The heating effect is proportional to the square of the current, and hence this type of instrument indicates the average value of the square of the current. If the positions of the pointer as assumed for various values of a steady direct current are marked, that is, if the instrument is calibrated by direct current it may then be used to indicate root-mean-square or effective values of an alternating current.

The thermocouple is a hot-wire instrument where, instead of indicating the heating effect as above, the heat liberated by the conductor carrying the current is allowed to act upon a thermo-electromotive element, that is, a junction point formed by two dissimilar conductors. The electromotive force thus produced may be used to cause a motion of a moving-coil instrument. In the so-called thermo-galvanometer the moving coil of the galvanometer contains a thermocouple and is suspended immediately above the heating coil.

Of the two devices so far considered, one averages the current and the other averages the square of the current. The former is not adaptable to the detection and measurement of alternating currents, while the latter is so adaptable.

Methods and Means of Detection.—We are now ready to consider in general terms the possible methods of detecting alternating currents and the means employed. These will be

classified first as to the sense affected, thus: (1) methods applicable to visual indications; (2) methods applicable to audible indications. In both cases the current must produce a motion of some part of a mechanical system. In the first case this motion must be slow enough to be visible, and in the second case it must be periodic and fast enough to be audible, but not so fast as to be beyond the audible limits. Of course, a motion which would be slow enough in its alternations to be visible, as for example that of a telegraph sounder, may also be used to give discrete audible sounds and is generally so employed. A sensitive ammeter in series with the ordinary telegraph sounder will give perfectly good visual indications, but the sounder is preferred as it leaves the eyes of the observer free for use in recording the received signals. The classification made above may be made more clear cut by grouping the motions as periodic and audible (2) or as aperiodic and either visible or audible (1). Under method (1) we place then those methods in which the alternating current is caused to produce a unidirectional mechanical force and under (2) those methods in which the current is caused to produce an alternating mechanical force of an audible frequency.

1. The unidirectional force may be produced: (*a*) directly, as in a hot-wire galvanometer; (*b*) by converting the current into a unidirectional current and allowing it to act on an ordinary direct-current instrument, as in the case of the complete rectification accompanying a synchronous commutation of the current by mechanical switching; (*c*) by causing the current to produce a unidirectional e.m.f. in an auxiliary circuit, as in the thermocouple; or (*d*) by causing the current to produce a change in the conductivity of an auxiliary circuit containing a steady unidirectional e.m.f., as in the case of coherers or of barretters.

2. The alternating mechanical force may be produced: (*a*) directly by the current, if of the proper frequency; or (*b*) a pulsating force may be produced, and hence an alternating motion of a receiver diaphragm may be obtained of any desired

frequency, if periodic interruptions are made by a switch in the alternating-current circuit and if this interrupted alternating-current is detected by any of the methods (1*b*), (1*c*) or (1*d*) above; or (*c*) the desired audible frequency may be obtained by producing so-called "beats" of the impressed alternating current with an auxiliary current, as will be described later.

It will be noticed in connection with the method (2*b*) that it is immaterial whether the interruptions in the alternating-current are occasioned at the source of the alternating-current (as for example, at the transmitting station in the case of a spark or quenched gap transmitting system) or are introduced at the detecting circuit (as for example, in the case of the use of a rotating switch or "tikker" in receiving continuous waves).

No attempt will be made to describe all the means by which these methods are applied, but illustrations will be given.

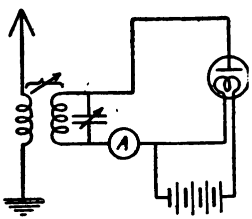


FIG. 29.—Fleming valve as a detector.

(1*a*) Electrodynamometers, plunger-type galvanometers and hot-wire galvanometers all give indications proportional to the square of the input current. Thus if the current is $I \sin \omega t$ the effect is $K \sin^2 \omega t$ or $K/2 - K (\cos 2\omega t)/2$. There is then a steady or average effect of $K/2$ which may be observed while the double-frequency effect is zero on the average.

(1*b*) The production of a undirectional current from an alternating current by mechanical commutation or switching is not practicable at very high frequencies. Such a system would give, of course, a complete rectification of the impressed current wave. A somewhat similar effect may, however, be produced by making use of a conductor which will transmit current in one direction only and thus transmits only alternate half waves of the current. As an illustration may be mentioned the Fleming valve which is the two-element vacuum tube of Fig. 19 with the battery E_B replaced by the source of the e.m.f. to be detected and by a gal-

vanometer, A, in series as in Fig. 29. If the input to the tube consists of an interrupted alternating current, then a receiver may be substituted for a galvanometer.

An incomplete suppression of the alternate half waves may be obtained by using a crystal detector as in Fig. 30 or Fig. 31. The detector D is formed by two dissimilar conductors in contact at some point. Thus steel and carborundum, steel and silicon contacts are used as detectors. Metallic contacts with pyrite or with galena are used as well as many other combinations such as zincite and chalcopyrite, some of which are known by trade names. All these combinations are found to have resistances

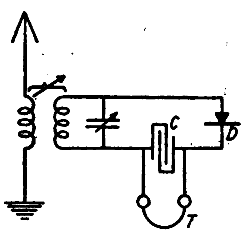


FIG. 30.—Circuit for a crystal detector.

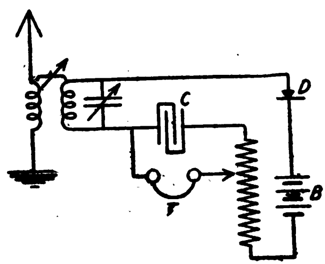


FIG. 31.—Crystal detector and superimposed direct current.

across the contact which depend for their values upon the e.m.f. applied to the contact. In general the resistance will be so much higher for an e.m.f. in one direction across the contact than for the same value of e.m.f. in the reversed direction that they are practically unilateral in conductivity. Thus Fig. 32 shows a typical current-voltage curve. It is evident, then, that if such a detector were connected as in Fig. 30 small e.m.f.'s impressed across it would meet practically infinite resistance. If, however, the detector is connected as in Fig. 31 and a voltage of oa (see Fig. 32) is applied to it, then it will be sensitive to smaller inputs and will transmit one half wave more efficiently than the other. In this form it operates much like a vacuum tube as shown in

Fig. 25 when E_c is Oa. It is, however, much less sensitive and is incapable of producing an amplification.

The condenser C is inserted in shunt with the receiver T, in Figs. 30 and 31, so as to offer a path of low impedance to the high-frequency current which is to be detected.

(1*c*) Thermocouples and magnetic detectors which form this class are of more interest historically then practically today.

(1*d*) The same is true of coherers and electrolytic detectors.

(2*a*) The production of an audible note by a telephone receiver when an alternating e.m.f. of audible frequency is impressed on the winding has been discussed in Chapter II.

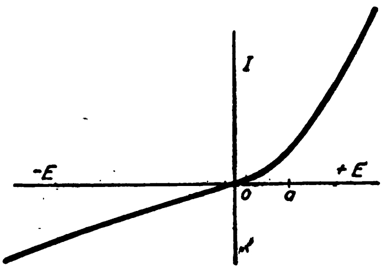


FIG. 32.—E.m.f. current characteristic of crystal detector.

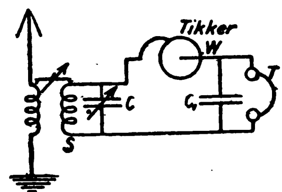


FIG. 33.—Tikker circuit.

(2*b*) Figs. 30 and 31 illustrate this method. For the circuits shown the interruptions are produced at the sending station. If they were not so produced it would of course be necessary to insert an interrupter in the detector circuit.

Alternating currents may, however, be detected by the use of an interrupter without using any of the detectors classified in (1*b*) to (1*d*) above. An arrangement for doing so is shown in Fig. 33. In this figure W is a rotating switch or contact-maker called a "tikker" when used in this way. It alternately connects and disconnects C_1 in parallel with C. When C_1 is connected it receives a charge from the coil S or, more exactly, from C. When C_1 is disconnected it is allowed to discharge through the telephone

receiver. Each time W disconnects C_1 from C there is a click in the receiver, if there is an alternating current in S. These clicks are controlled in frequency of occurrence by the speed of rotation of the switch and may thus be caused to produce an audible note in the receiver.

In the case just considered the tikker makes its contact for a length of time large as compared to the period of the alternating-current to be detected. In the case of the so-called "tone wheel" the speed of the rotating switch is very high and the frequency of its interruptions of the circuit may be made that of the alternating current. In this case the circuit is that of Fig. 33 except that the condenser C_1 is omitted. Suppose the wheel runs at the same frequency as the alternating current; then it will allow the same portion of the alternating-current wave to be impressed on the receiver each time it makes contact. There will then be no sound in the receiver since the alternating current is well above audible frequency limits. If, however, the wheel runs a little faster (or a little slower) then it will impress on the receiver a slightly different part of the wave form each time it makes the contact until it has impressed all the different parts of the wave and is ready to impress again the same part as it did at the moment when its action was first considered. This cycle will occur $f_w - f$ times a second if f_w and f are the frequencies of the wheel and the alternating current respectively. There will then be produced cyclic variations in the e.m.f. impressed on the receiver at a frequency, $f_w - f$, which may be controlled and made audible by regulating the speed of the wheel.

(2*c*) To study the method of detecting an alternating current by audible beats it is best first to consider the effect of combining two currents of different frequencies. Thus let the two currents be of maximum amplitude C and B, respectively, and of frequencies $f_0 = \frac{\omega_0}{2\pi}$ and $f_0 + f_1 = f_0 + \frac{\omega_1}{2\pi}$.

Then assume the currents to be $C\epsilon^{j\omega_0 t}$ and $B\epsilon^{j(\omega_0+\omega_1)t}$. The sum i of these currents is

$$
\begin{aligned}
i &= C\epsilon^{j\omega_0 t} + B\epsilon^{j\omega_0 t}\cdot\epsilon^{j\omega_1 t} \\
&= (C + B\epsilon^{j\omega_1 t})\epsilon^{j\omega_0 t} \\
&= A\epsilon^{j\omega_0 t} \quad \text{where } A = C + B\epsilon^{j\omega_1 t}
\end{aligned}
$$

This current is evidently a sinusoidal current of frequency f_0, the amplitude A of which varies with a frequency of f_1. The maximum value of A is $C + B$ and the minimum is $C - B$, corresponding to $\omega_1 t = 0, 2\pi$, etc., and $\omega_1 t = \pi$, 3π, etc., respectively. In Fig. 34 are illustrated typical relations for two different value of t.

Now it is evident that if these two currents pass through the same telephone receiver they will give rise to an alternat-

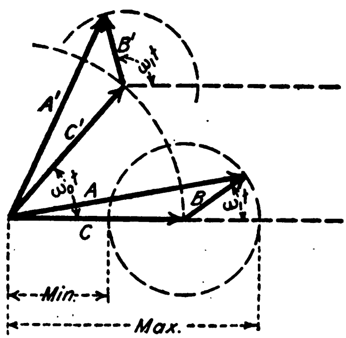


FIG. 34.—Combination of two e.m.fs. of different frequencies.

ing motion of the diaphragm at a frequency of f_0. The maximum amount of this alternating motion will, however, vary from time to time, being also periodic with a frequency of f_1. If f_0 is an audible frequency then the listener will hear the note corresponding to f_0 wax and wane f_1 times per second.

If, however, f_0 is above audibility the note can not be heard, and hence, of course, the variations in its intensity can not be observed no matter what their frequency may be. That is, it is not possible by the combination of two sinusoidal currents which are above audible frequency to produce an audible frequency in a translating device (*e.g.*, a telephone receiver) which repeats without distortion.

Suppose, however, that the distortion in the receiver is made appreciable, as for example by removing the permanent magnet so that the pull becomes $kk_1^2i^2$.

For this case we must write the conjugate vectors, thus:

$$i = C\epsilon^{j\omega_0 t} + C\epsilon^{-j\omega_0 t} + B\epsilon^{j(\omega_0+\omega_1)t} + B\epsilon^{-j(\omega_0+\omega_1)t}$$

Hence

$$\begin{aligned} i^2 = {} & C^2\epsilon^{j2\omega_0 t} + C^2\epsilon^{-j2\omega_0 t} + B^2\epsilon^{j2(\omega_0+\omega_1)t} + B^2\epsilon^{-j2(\omega_0+\omega_1)t} \\ & + 2CB\epsilon^{j\omega_0 t}.\epsilon^{j(\omega_0+\omega_1)t} + 2CB\epsilon^{-j\omega_0 t}\epsilon^{-j(\omega_0+\omega_1)t} + 2C^2 \\ & + 2CB\epsilon^{j\omega_0 t}\epsilon^{-j(\omega_0+\omega_1)t} + 2CB\epsilon^{-j\omega_0 t}\epsilon^{+j(\omega_0+\omega_1)t} + 2B^2 \end{aligned}$$

or

$$\begin{aligned} i^2 = {} & C^2(\epsilon^{j2\omega_0 t} + \epsilon^{-j2\omega_0 t}) + B^2(\epsilon^{j2(\omega_0+\omega_1)t} + \epsilon^{-j2(\omega_0+\omega_1)t}) \\ & + 2CB(\epsilon^{j(2\omega_0+\omega_1)t} + \epsilon^{-j(2\omega_0+\omega_1)t}) \\ & + 2CB(\epsilon^{j\omega_1 t} + \epsilon^{-j\omega_1 t}) + 2C^2 + 2B^2 \end{aligned} \tag{37}$$

It will then be observed that there results a term of double the frequency f_0, a term of double the frequency $f_0 + f_1$, a term of

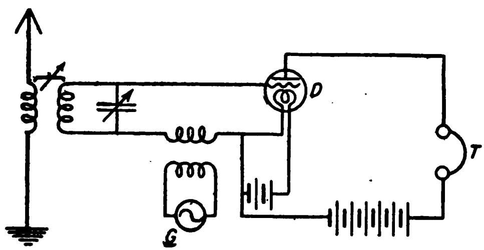


FIG. 35.—Heterodyne receiving system.

frequency the sum, namely $2f_0 + f_1$ and one of frequency the difference, or f_1. Of these terms only the last can cause an audible vibration of the receiver diaphragm. If f_1, the difference between the two frequencies, is within the audible limit then their combination in a device following the square law will result in an audible note. The terms $2C^2$ and $2B^2$ are of zero frequency.

This method of producing an audible note from an inaudibly high frequency is known as the heterodyne method. Fig. 35 shows a circuit arrangement for applying this method, where G is an auxiliary generator of frequency higher or lower by f_1 than the

frequency f_0 which is to be detected. Any device following the square law may be used to produce the desired audio-frequency "beat note" from the two impressed e.m.f.'s. The sketch, however, shows a vacuum tube.

To apply this heterodyne method and to obtain an audible note may be accomplished by devices other than those following the square law. Any detector may be used. Now in general terms any device for which the output is not directly proportional to the input may be used as a detector, that is, any device for which the relation between output and input is other than "linear."

Vacuum-tube Detector.—The vacuum tube has not been placed in the classification of detectors as made above. Its characteristics are such that it might be classified in any of several of the groups indicated. Thus it might be considered to act as in (1*b*) as a partial rectifier, as in (1*c*) as causing a unidirectional e.m.f. in the output circuit, or as in (1*d*) as a device changing the resistance of a circuit, in this case its own output circuit. Since it is capable of broader use than any of the other detectors it seems best to let it form a type by itself.

CHAPTER V

THE PRODUCTION OF DAMPED SINUSOIDAL CURRENTS

Damped Oscillations.—If a pendulum is started swinging and allowed to die down the curve showing its displacement at each instant of time will be of the form shown on the right-hand side of Fig. 36. Now it is obvious that this is the curve given by the vertical component of a rotating vector which is constantly shrinking in length as shown on the left-hand side of the figure. Suppose the rotating vector to be $A' = A_0\epsilon^{j\omega t}$, then the constant shrinking can be indicated by multiplying A' by ϵ^{-at}. That is, the decreasing vector A, as given by

$$\begin{aligned} A = A_0\epsilon^{-at}\epsilon^{j\omega t} &= A_0\epsilon^{-at}(\cos \omega t + j \sin \omega t) \\ &= A_0\epsilon^{-at}\cos \omega t + jA_0\epsilon^{-at}\sin \omega t \end{aligned}$$

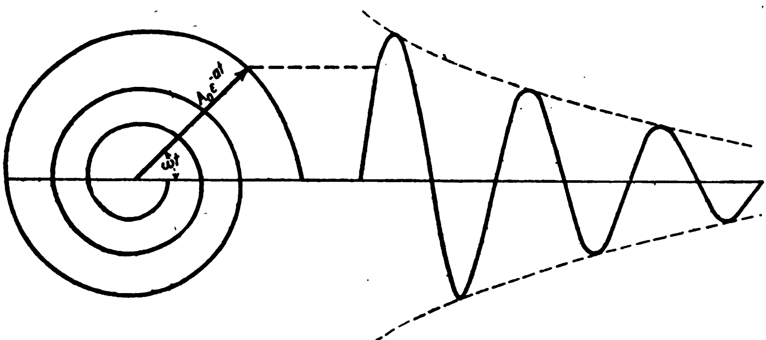


FIG. 36.—Vector representation of a damped sinusoid.

is the sum of two vectors, one along the axis of reals and the other along the axis of imaginaries. For the moment we shall consider only the vector $jA_0\epsilon^{-at}\sin \omega t$ whose variations are shown in the figure.

To see more clearly how this vector varies let us consider the maximum amplitudes A_1 and A_2 of any two successive swings in the same direction by giving ωt the values below.
If

$$\omega t = \omega t_1 \text{ then } t = t_1$$

and

$$A_1 = A_o \epsilon^{-at_1} \epsilon^{j\omega t_1}$$

If

$$\omega t = \omega t_2 = \omega t_1 + 2\pi \text{ then } t_2 = t_1 + 2\pi/\omega$$

and

$$\epsilon^{-at_2} = \epsilon^{-a(t_1 + 2\pi/\omega)}$$

hence

$$A_2 = A_0 \epsilon^{-at_2} \epsilon^{j\omega t_2} = A_0 \epsilon^{-at_1 - 2\pi a/\omega} \epsilon^{j\omega t_1 + j2\pi}$$

Then

$$\frac{A_2}{A_1} = \epsilon^{\frac{-2a\pi}{\omega}} \epsilon^{j2\pi} = \epsilon^{\frac{-2a\pi}{\omega}}$$

The ratio A_2/A_1 is evidently independent of the time t_1, hence this ratio also represents the ratio of any displacement to the displacement which preceded it by a cycle. The value of this ratio is called the damping. The logarithm of the damping to the base ϵ is defined as the logarithmic decrement.[1] The logarithmic decrement d is then

$$d = \log A_2/A_1 = \log \epsilon^{2a\pi/\omega} = \frac{2a\pi}{\omega} = \frac{2a\pi}{2\pi f} = \frac{a}{f} \tag{38}$$

If the decrement d and the frequency f are known then the factor a is found as df.

The trace shown in Fig. 36 happens actually to be a copy of an oscillogram of a highly damped alternating current and not the motion of a pendulum. In the first chapter we were concerned only with steady or sustained alternating e.m.f.s and

[1] Damping and logarithmic decrement are sometimes given per half period, but standard practice is to define them in terms of a whole period. The damping per whole period is obviously the square of the damping per half period and the logarithmic decrement per whole period is twice that for the half period.

currents. In the light of the discussion just preceding we see then that we may express damped sinusoidal functions such as currents or e.m.f.s by means of rotating vectors and an exponential term representing the damping. The vector $A_0\epsilon^{-at}\epsilon^{j\omega t} = A_0\epsilon^{(-a+j\omega)t}$ represents, however, as stated above the sum of two vectors, one along the axis of reals and the other along the axis of imaginaries. To represent a real damped sinusoidal current, as i, it is necessary then to write the sum of a pair of conjugate vectors,

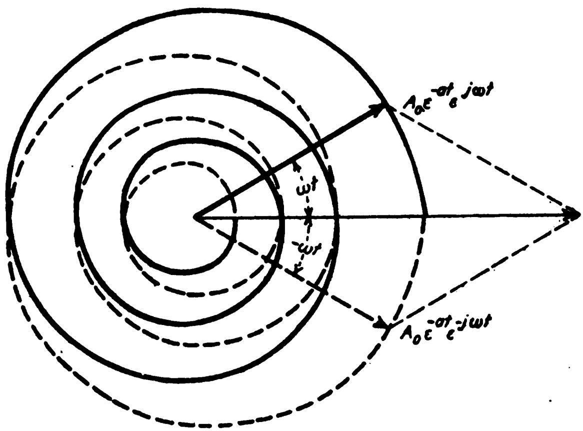


FIG. 37.—Conjugate vectors representing damped sinusoid.

thus

$$i = I\epsilon^{(-a+j\omega)t} + \bar{I}\epsilon^{(-a-j\omega)t} \tag{39}$$

This is, a rigorous and general expression for an exponentially damped sinusoidal current just as equation (14) of page 16

$$i = I\epsilon^{j\omega t} + I\epsilon^{-j\omega t} \tag{14'}$$

is the expression for a sustained sinusoidal current of the same angular velocity ω. In Fig. 37 are shown two conjugate vectors, exponentially damped, and their sum, at one definite instant.

General Form for Representing a Current or E.m.f.—The expression of equation (39) above is a perfectly general expression for a current. Thus if a is zero the current i is a sustained sinusoid of frequency $\omega/2\pi$. If ω is zero the current i does not alternate and hence is a direct current. If both a and ω are zero the current i is a steady direct current. If $-a$ is positive the current is building up or increasing as time progresses. If $-a$ is negative the current is decaying.

If a current is composed of several component frequencies, that is, of several harmonics, each component may be represented as in equation (39) and the total current will be represented by the sum of these separate components. In this form it is very easy to obtain the rate of change of a current of a complicated wave form, since its rate of change will be the sum of the rates of change of its several components.

Decaying Currents.—The current in a circuit damps down or dies out when the e.m.f. which established it is removed. If the circuit contains one or more storage reservoirs of energy this decay may be very gradual, a current being maintained until the energy of the reservoir has been dissipated in heat in the resistance of the circuit.

Oscillating Circuit.—If both types of storage reservoirs, that is, inductance and capacity, are contained in the circuit, then as one gives up energy the other will absorb it. Such a circuit is called oscillating for the reason that the current surges back and forth as a result of storing the energy first in the condenser and then in the inductance. The current alternates in direction with a frequency determined, as will be shown, by the values of the inductance and the capacity (and to some extent by the resistance).

Thus if a condenser which has been charged by a battery is connected to an inductance, the condenser starts to discharge through the conducting circuit offered to it by the inductance. The condenser sends a discharge current through the circuit, losing the energy of its electrical field, but storing

up energy in the magnetic field of the inductance. As soon, however, as the discharge current ceases to increase, the magnetic field of the inductance starts to collapse, and then the e.m.f. of self-induction maintains a decreasing current in the same direction. The current from the discharging condenser, while increasing, has reduced to zero the excess of electrons on the negative plate of the condenser, and a further current or stream of electrons in the same direction must result in this plate having a deficiency of electrons, that is, becoming charged positively or in the opposite direction. Thus the effect of the flow of current during the collapse of the magnetic field of the inductance is to charge the condenser again, but in the opposite direction. When the current due to the inductance has at last ceased, the condenser is ready again to discharge, but evidently in the opposite direction. This alternate charge and discharge of the condenser, first in one direction and then in the opposite direction, goes on indefinitely except in so far as energy is dissipated and taken out of the circuit in the form of heat. If the connecting wires were of zero resistance and, what is still more impossible, if the inductance coil had zero resistance, then the oscillations would, of course, continue indefinitely. But energy is dissipated all the time, for the current must flow through some resistance; hence, since the capacity remains the same, the potential to which the condenser is charged, and the resulting current, gradually decrease. The current is then said to be damped out by the resistance.

Transient Currents.—If only one type of reservoir is contained in the circuit there are no alternations and the current gradually dies down to zero. Such a current is to be considered as a damped sinusoidal current of zero frequency.

Similarly, as is well known, the current due to an e.m.f. impressed upon a circuit containing inductance builds up slowly to its full value, for energy must be stored in the inductance. Such a building-up current may for convenience be considered to be the sum of two currents, a steady current, I_0, which will ultimately

flow when conditions have become stable and a transient current, $I\epsilon^{-at}$ called into play by the change in the impressed e.m.f. from zero (just before it is applied) to the actual value applied. The actual current which flows in the circuits starts from zero at the moment when the e.m.f. is applied. That is, when $t = 0$, $i = I_0 + I\epsilon^{-a0} = 0$, or since $\epsilon^{-a0} = 1$ we have $I_0 = -I$. Hence the transient current is to be thought of as having, at $t = 0$, a value equal and opposite to the current I_0 which the e.m.f. will ultimately establish. As t increases $I\epsilon^{-at}$ decreases and hence the sum $I_0 - I_0\epsilon^{-at}$ approaches its steady state value of I_0. The steady current which will ultimately flow is called the "forced current."

It is convenient in the case of decaying currents to consider that when the e.m.f. applied to a circuit is reduced to zero it is the result of impressing an e.m.f. equal and opposite to that already active. Thus imagine an inductive circuit in which a sustained e.m.f. E is acting and a sustained current of I_0 is flowing. Let an e.m.f. of $-E$ be impressed. The total e.m.f. acting on the circuit is now zero. The current which will ultimately flow is also zero. But there accompanies the impression of the e.m.f., $-E$, a transient current $I\epsilon^{-at}$, such that the total current $-i$ due to $-E$ is $-i = -I_0 + I\epsilon^{-at}$. That is, the transient current starts with a value $I = I_0$. In other words, when the e.m.f. in the circuit is reduced to zero a decaying transient current is called into play and this current starts with a value equal to the current I_0 which has been flowing in the circuit.

The conditions for the two cases are tabulated below:

	Initial current	Forced current	Transient current
At $t = 0$; E applied	0	I_0	$-I_0\epsilon^{-at}$
E already active	I_0	I_0	
$-E$ applied	....	$-I_0$	$I_0\epsilon^{-at}$
Net sum	....	0	

In general then we shall represent the current flowing in a cir-

cuit as the sum of two currents, one the forced current which the e.m.f., will ultimately cause to flow and the other a transient current or current natural to the circuit since its wave form will depend only upon the circuit. In the case just considered only one type of reservoir is in the circuit and the transient current does not alternate, that is, it is of zero frequency. If both types of reservoirs are present the transient will be a decaying sinusoid of the form $I_0\epsilon^{(-a+j\omega)t}$. Hence we shall speak of such transients as the natural (current) oscillations remembering that their frequencies may be zero. The transient current or natural oscillation current of a circuit is then a current which depends upon the circuit and accompanies any change in the impressed e.m.f., as for example, a transient occurs when an e.m.f. is applied and also when it is removed.

The transient current accompanying the application of an e.m.f. is perhaps a fictitious current, but it is of great convenience to consider the actual current a sum of two fictitious currents, one of which follows the variations in the driving e.m.f., and the other of which is a natural oscillation in the circuit. On the other hand, there is no mathematical fiction to the transient which accompanies the removal of an impressed e.m.f. Thus the curve of Fig. 36 was taken in the laboratory by an oscillograph, connected in a circuit containing an inductance with resistance and a capacity. The condenser was charged by a battery in series with the circuit. The battery was then disconnected and replaced by a short circuit. Following the removal of the battery there was a current in the circuit, as shown by the oscillogram of which part is reproduced in the figure.

Application of Transients to Wireless Operation.—It is these transient oscillations that are made use of in spark-gap transmitting systems for wireless. Thus consider Fig. 38 where is shown the system for transmitting wireless as used by Marconi in 1896. A battery B which is controlled by a key K is connected to the primary winding of an induction coil I. This induction coil induces in its secondary winding an e.m.f. of much higher voltage

than the battery, due to the sudden opening of the primary circuit by an interrupter S. Each time the interrupter acts a high voltage is impressed on a gap G formed by two metal spheres. This air gap is broken down when the voltage is high enough

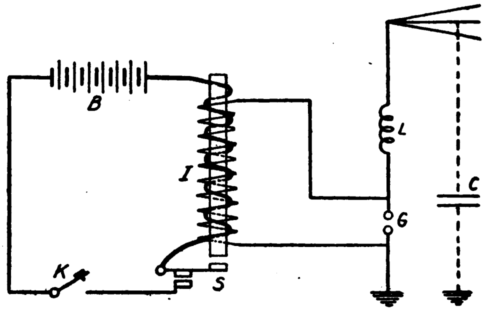


FIG. 38.—Marconi transmitter of 1896.

and forms a conducting path after the manner described on pages 39 and 40. Before the spark passes across G the voltage from the coil is charging up the condenser, formed by the aerial and the earth, which is indicated in Fig. 38 at C by dotted lines. When the gap discharges it offers a path of low resistance for the e.m.f. of the induction coil and also for the discharge of C. Now there is also some inductance in the aerial circuit and this has been indicated by L in the figure. There results then every time the spark gap breaks down a state of affairs similar to that obtaining for the curve of Fig. 36. It is desirable, therefore, to determine the damping and the frequency of the natural oscillation of such a circuit.

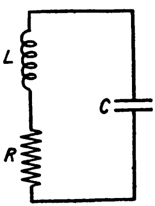


FIG. 39.—Equivalent circuit for Fig. 38.

Frequency Constants of a Circuit of One Degree of Freedom.—The circuit used by Marconi consists of inductance, capacity, and resistance in series as shown in Fig. 39. Imagine an e.m.f. of v inserted in this circuit and that a current i flows as a result.

The impressed e.m.f. must equal the sum of the e.m.f.'s necessary to force this current through R, L and C, respectively, that is,

$$v = Ri + Lpi + p^{-1}\, i/C \qquad (40)$$

Since the transient current will be a damped sinusoid put $i = I\epsilon^{jxt}$ where $jx = -a + j\omega$.
Then

$$v = RI\epsilon^{jxt} + LjxI\epsilon^{jxt} + \frac{I\epsilon^{jxt}}{jxC}$$

Thus if v is made zero, we have, since i is not zero,

$$R + jxL + \frac{1}{jxC} = 0 \qquad (41)$$

or

$$j^2x^2LC + jxRC + 1 = 0$$

Solving this for jx we obtain

$$\begin{aligned} jx &= \frac{-RC \pm \sqrt{R^2C^2 - 4LC}}{2LC} \\ &= -\frac{R}{2L} \pm \frac{1}{\sqrt{LC}}\sqrt{\frac{R^2C}{4L} - 1} \\ &= -\frac{R}{2L} \pm \frac{j}{\sqrt{LC}}\sqrt{1 - \frac{R^2C}{4L}} \qquad (42) \\ &= -a \pm j\omega \end{aligned}$$

From this equation it is evident that in a circuit containing R, L and C the frequency of the natural oscillation depends only upon L and C, provided that $4L$ is large as compared to R^2C. The damping constant is directly proportional to the resistance, and inversely proportional to the inductance.

Special Property of the Operator "p."—Use has just been made in determining the e.m.f. across the inductance and the condenser of Fig. 39 of the fact that if $i = I\epsilon^{jxt}$ then $pi = jxI\epsilon^{jxt}$. Suppose that for jx we write the letter p', then

$$i = I\epsilon^{p't} \quad \text{and} \quad pi = p'I\epsilon^{p't} = p'i$$

In other words, if we let p' stand for jx, or in general for $-a+j\omega$, then we find that the rate of change of the current (symbolized as pi) is $p'i$ where p' is a complex number. If then in equation (40) namely

$$v = Ri + Lpi + \frac{p^{-1}i}{C} \tag{40'}$$

we substitute

$$i = I\epsilon^{p't}$$

we obtain

$$v = Ri + Lp'i + \frac{i}{p'C} \tag{43}$$

which we may solve algebraically for p' and thus obtain jx. But except for the primes in equation (43) the equations (40′) and (43) are identical. We see then that if we write $i = I\epsilon^{pt}$, we may treat the operator p of equations like (40) as an ordinary algebraic quantity.

Symbolic Impedance.—In equation (40) if the operator p is treated in this way then the term i may be factored out, thus

$$v = (R + Lp + p^{-1}/C)i$$

Hence

$$Z = \frac{v}{i} = R + Lp + \frac{p^{-1}}{C} \tag{44}$$

The right-hand side of equation (44) is a symbolic expression for the impedance of the series circuit of Fig. 39. To determine the natural frequency constants of the circuit we put the symbolic impedance equal to zero and solve for p, in which case we obtain $p = a \pm j\omega$ where a and ω have the values previously found in equation (42). Similarly if the impedance to an impressed e.m.f. of frequency constants, $a + j\omega$, is desired it may be found by substituting this value of p in the symbolic impedance. Thus if a sustained e.m.f. of frequency $\omega_1/2\pi$ is impressed, that is if $v = E\epsilon^{(0+j\omega_1)t} = E\epsilon^{j\omega_1 t}$, the impedance is $Z = R + Lj\omega_1 + \dfrac{1}{j\omega_1 C}$ which is the same value as will be found in Problem 4 of page 187.

Conjugate Frequency Constants.—In the preceding solutions for the frequency constants of the circuit of Fig. 39 a quadratic equation was solved giving two values, $-a + j\omega$, and $-a - j\omega$, as in equation (42). The explanation of the occurrence of this pair of conjugates lies of course in the fact that the current which flows in the circuit is a real current and therefore is not to be represented as $i = I\epsilon^{jxt}$ but as $i = I\epsilon^{jxt} + \bar{I}\epsilon^{-jxt}$ or as $i = I\epsilon^{p_1t} + \bar{I}\epsilon^{p_2t}$ where p_1 and p_2 are conjugates. Considering the last form of expression for the current we have

$$Lpi = Lp_1I\epsilon^{p_1t} + Lp_2\bar{I}\epsilon^{p_2t}$$

$$\frac{p^{-1}i}{C} = \frac{I\epsilon^{p_1t}}{p_1C} + \frac{\bar{I}\epsilon^{p_2t}}{p_2C}$$

$$Ri = RI\epsilon^{p_1t} + R\bar{I}\epsilon^{p_2t}$$

Hence

$$v = Ri + Lpi + \frac{p^{-1}i}{C}$$

$$= \left(R + Lp_1 + \frac{1}{p_1C}\right)I\epsilon^{p_1t} + \left(R + Lp_2 + \frac{1}{p_2C}\right)\bar{I}\epsilon^{p_2t}.$$

If v is zero then since i is not zero we have

$$R + Lp_1 + \frac{1}{p_1C} = 0 \tag{45}$$

and

$$R + Lp_2 + \frac{1}{p_2C} = 0 \tag{46}$$

These two equations are obviously satisfied by substituting for either p_1 (or p_2) the value $-\frac{R}{2L} + \frac{j}{\sqrt{LC}}\sqrt{1 - \frac{R^2C}{4L}}$ and then for p_2 (or p_1) the conjugate.

In problems of the type under consideration it is, therefore, convenient to follow the method of page 22 and to write but one conjugate vector of the pair. In that case it is usual to deal

with the counter-clockwise rotation by using the solution involving the positive value of j, thus $-a + j\omega$.

Circuits of More than One Degree of Freedom.—If two pendula are isolated and are set swinging, each swings with its own natural frequency. If, however, they are coupled together, as by being hung from a common elastic support as in Fig. 40, then they react upon each other and the motion of neither pendulum is what it would be if alone. If the trace is made of the motion of one of the pendula it will be found to be the resultant of two sinusoidal motions of different frequencies. If the motion of the other pendulum is analyzed it also will be found to be the resultant of two sinusoids of the same frequencies as in the case of the first pendulum. That is, the displacement

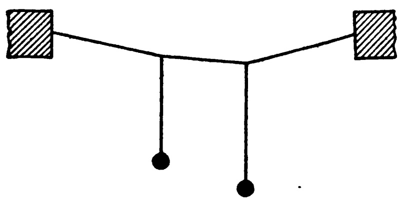


Fig. 40.—Mechanical system of two degrees of freedom.

of one of the pendulum bobs from its position of rest is no longer a simple sinusoidal function of the time but is the sum of two such sinusoids. In general terms[1] then its motion may be represented by $A\epsilon^{j\omega_1 t} + B\epsilon^{j\omega_2 t}$. The mechanical system composed of the two coupled pendula is then said to have two "degrees of freedom." In other words, it oscillates with two natural frequencies simultaneously. The single pendulum has but one degree of freedom and can have *natural* oscillations of only one period.

In the case of a piano or violin string the reader will remember

[1] The conjugates $\overline{A}\epsilon^{-j\omega_1 t}$ and $\overline{B}\epsilon^{-j\omega_2 t}$ are omitted for convenience and any phase difference is cared for by the vectors A and B after the manner of equations 21 and 22.

that the fundamental note may be produced and at the same time one or more overtones. Such a string consists in effect of an infinite number of small particles elastically connected and hence has an infinite number of degrees of freedom. In the case of the violin string the various possible frequencies of vibration are simply related, all being multiples of the lowest frequency. In the case of the pendula and of the electrical circuits which will now be examined no such simple relation as fundamental and harmonics exists.

Consider for example the circuit of Fig. 41. Let an e.m.f. v be active in one branch as shown. Let the unknown currents in the two branches be represented by i_1 and i_2 in the directions assumed. The values of the inductance and capacity are as

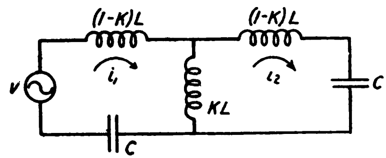


FIG. 41.—Electrical system of two degrees of freedom.

indicated, the inductance KL being common to the two branches. For simplicity the resistance is assumed to be negligible. Now the voltage v impressed in branch 1 must be equal to the e.m.f.'s across the inductance $(1 - K)L$, that is $(1 - K)Lpi_1$, plus that across KL and C. The current through KL is $i_1 - i_2$, hence this e.m.f. is $KLp(i_1 - i_2)$, that is $KLpi_1 - KLpi_2$. The e.m.f. across the condenser is $(p^{-1}i_1)/C$. Then the value of v is

$$\begin{aligned} v &= (1 - K)Lpi_1 + KLp(i_1 - i_2) + \frac{p^{-1}i_1}{C} \\ &= Lpi_1 + \frac{p^{-1}i_1}{C} - KLpi_2 \\ &= \left(Lp + \frac{p^{-1}}{C}\right)i_1 - (KLp)i_2 \qquad (47) \end{aligned}$$

which is in the form $v = ai_1 - bi_2$ where a and b are symbolic

impedances of the values shown in the equation. In the second branch there is no impressed e.m.f., hence for that branch we have

$$0 = (1 - K)Lpi_2 + \frac{p^{-1}i_2}{C} + KLp(i_2 - i_1)$$

$$= -(KLp)i_1 + \left(Lp + \frac{p^{-1}}{C}\right)i_2 \tag{48}$$

which is in the form $0 = -bi_1 + ai_2$ where a and b are as above. Solving simultaneously the two equations to determine i_1 gives

$$v = \frac{a^2 - b^2}{a} i_1$$

Hence upon substituting for a and b we have

$$Z = \frac{v}{i_1} = \frac{\left(Lp + \frac{1}{Cp}\right)^2 - (KLp)^2}{\left(Lp + \frac{1}{Cp}\right)} \tag{49}$$

where Z is in symbolic form the impedance which the network of Fig. 41 offers to an e.m.f. of v impressed in branch 1 as shown.

If this impedance is put equal to zero we find by solving for p the frequency constants of the natural oscillations of the branched circuit. Multiplying both sides of equation (49) by $\left(Lp + \frac{1}{Cp}\right)$ and placing $Z = 0$ gives

$$(1 - K^2)L^2p^2 + \frac{2L}{C} + \frac{p^{-2}}{C^2} = 0$$

or

$$(1 - K^2)L^2C^2p^4 + 2LCp^2 + 1 = 0$$

hence

$$p = \pm \frac{j}{\sqrt{LC}\,\sqrt{1 - K}} \tag{50}$$

and

$$p = \pm \frac{j}{\sqrt{LC}\,\sqrt{1 + K}} \tag{51}$$

There are thus found two pairs of conjugate frequency constants. The damping factors are zero for both natural oscillations in this impractical case where the resistance is assumed zero.

Coupling.—The factor K is the coupling between the two branches of the circuit. In this case there is a conductive coupling. Where the branches are alike, K is the ratio of the inductance common to the two branches to the total inductance of either branch. In case the total inductance of one branch, say L_1, is not equal to that of the other, say L_2, then the coupling K is the square root of the ratio M^2/L_1L_2 where M is value of the inductance common to the two branches. Thus

$$K^2 = \frac{M^2}{L_1L_2} \tag{52}$$

This common or mutual inductance may be obtained by an inductive connection if there is a transformer connecting the two circuits. In this case M is defined by the equation

$$v_{12} = Mpi_2 \tag{53}$$

or

$$v_{21} = Mpi_1$$

where v_{12} is the voltage induced in circuit 1, by a current in circuit 2, which is changing at the rate pi_2. This defining equation is analogous to that for self-inductance, namely

$$v_{11} = Lpi_1$$

Impedances of Coupled Circuits.—Consider Fig. 41 again. When an e.m.f. is impressed at v we may be interested in the current i_1 which it causes to flow in its own branch or in the current i_2 which it causes in the branch coupled to it. In general terms, we have defined an impedance as the ratio of an e.m.f. to a resulting current. There may therefore be considered to be two impedances in this circuit, namely v/i_1 and v/i_2. Of these the first may be called the "driving-point impedance", considering v as a driving force and noting that i_1 is the resulting current at the point in the circuit where the

driving force is impressed. Similarly the second branch may be considered as driven by the e.m.f. impressed in the other branch. The ratio v/i_2, that is the ratio of the driving e.m.f. to the current at the driven point in the circuit, is called the "driving-driven point impedance."

Driving-point Impedance for Coupled Circuits.—Thus, for example, consider Fig. 42 where is shown an antenna to which is coupled a tuned circuit. Assume the constants to be as shown in the equivalent circuit of Fig. 43. Assume a voltage of v_1 active in the antenna, and assume currents as shown in

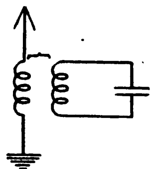


FIG. 42.—Practical radio system of two degrees of freedom.

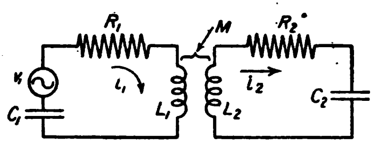


FIG. 43.—Equivalent circuit of Fig. 42.

the figure. Then the equations, which are similar to those of pages 75–76 become

$$v_1 = \left(L_1p + R_1 + \frac{1}{pC_1}\right)i_1 - (Mp)i_2 \tag{54}$$

$$0 = -(Mp)i_1 + \left(L_2p + R_2 + \frac{1}{pC_2}\right)i_2 \tag{55}$$

These are in the form

$$v_1 = a_1i_1 - bi_2$$

$$0 = -bi_1 + a_2i_2$$

Hence

$$Z_{11} = \frac{v_1}{i_1} = \frac{a_1a_2 - b^2}{a_2} \tag{56}$$

and

$$Z_{12} = \frac{v_1}{i_2} = \frac{a_1a_2 - b^2}{b} \tag{57}$$

If the frequency constants of the e.m.f. are known, thus if $v_1 = E\epsilon^{pt} = E\epsilon^{j\omega t}$, then upon substitution of $j\omega$ for p in equations (54) and (55) the driving-point impedance may be obtained by a further substitution in equation (56). Similarly for the driving-driven point impedance of equation (57).

For the special case where the antenna and its coupled circuit are both tuned to the frequency of the impressed wave train we have

$$\omega^2 = \frac{1}{L_1C_1} = \frac{1}{L_2C_2}$$

$$a_1 = R_1,\ a_2 = R_2,\ \text{and}\ b = jM\omega$$

hence

$$Z_{11} = \frac{R_1R_2 - j^2M^2\omega^2}{R_2} = R_1 + \frac{M^2\omega^2}{R_2} \tag{58}$$

Thus it is seen that in the case of an inductively coupled tuned receiving circuit such as is shown in Fig. 42 the resistance met by an undamped wave train of the frequency of the tuning is the sum of the antenna resistance and a resistance introduced by the coupled circuit.

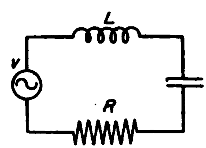


FIG. 44.—Simple tuned circuit.

Resonance Curves.—If an undamped sinusoidal e.m.f. of frequency constant ω is impressed on the circuit of Fig. 44 it will meet an impedance Z where

$$Z = \left(Lp + R + \frac{p^{-1}}{C}\right)$$

$$= R + \frac{j(LC\omega^2 - 1)}{C\omega} \tag{59}$$

This impedance becomes a minimum when $\omega^2 = 1/LC$. The frequency, corresponding to ω, is known as the resonant frequency. By comparison with equation (42) of page 71 it will be seen that for circuits of this form, involving negligible damping, the resonant frequency is the same as the natural

frequency. For all other cases the natural frequency is less than the resonant frequency.

When the impressed e.m.f. has the frequency of resonance, the impedance is merely the resistance of the circuit. For all other values of the impressed frequency the impedance is composed of the resistance term and a reactance. The numerical value of

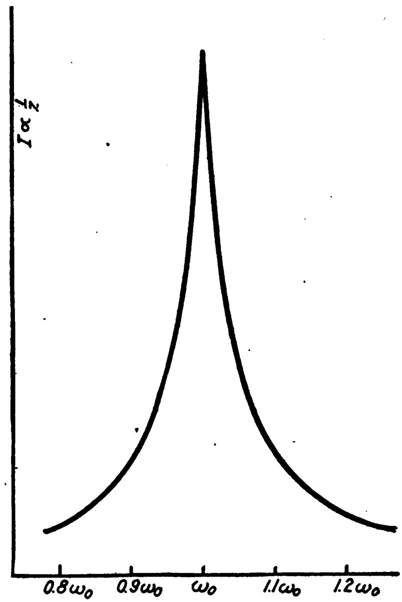


FIG. 45.—Resonance curve for Fig. 44.

this vector impedance is $Z = \sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}$. This has been shown to represent the ratio of the maximum value of the e.m.f. to the maximum value of the current. Since we are in general interested in the current it is convenient to plot against frequency the effective value of the current to be expected from e.m.f.'s of constant value but of different frequencies. In other words, it is convenient to make the plot shown in Fig. 45.

On the other hand, it is often convenient to know, in case the impressed e.m.f. is kept constant in both maximum amplitude and frequency, how the current will vary as the tuning of the circuit is altered. That is, we may vary the resonant frequency of the circuit by varying L or C and plot against the resonant frequency the reciprocal of the impedance which the circuit then offers to an impressed e.m.f. of constant frequency. This plot

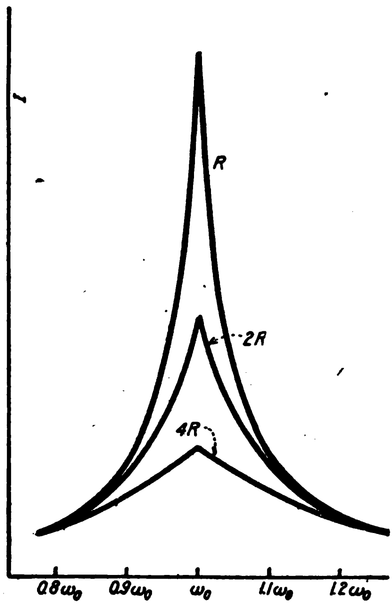


FIG. 46.—Effect on resonance of increased resistance.

is obviously identical with that of Fig. 45. Since this variation in resonant frequency is made by varying the capacity (or the inductance) it is customary to plot the current against the values of capacity (or inductance) instead of frequency. The abscissa of the resonance curve of Fig. 45 may then be L, C, or frequency as desired. Of course, when the abscissa is L, or C, the form of the curve will be changed somewhat from that of

Fig. 45 since the scale for abscissæ will be different for C and L which vary inversely as the square of the frequency.

If the resistance term of the impedance is large, then it is evident that the current is less affected by small changes in the value of the reactance term, that is the resonance curve is

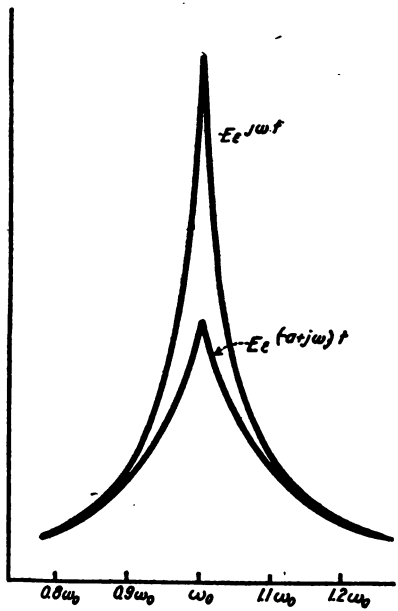


FIG. 47.—Resonance curve for a damped input.

flattened out by the introduction of resistance. Fig. 46 shows resonance curves for the circuit of Fig. 44 when R is changed from R to $2R$ and to $4R$.

Resonance Curves for Damped E.m.f.'s.—If the e.m.f. impressed on the circuit of Fig. 44 is not a sustained e.m.f. but is a damped e.m.f. of the form $v = E\epsilon^{(-a+j\omega)t}$ then the current i is found as $i = \frac{v}{Z}$ where Z is the impedance as given in equation (44). Thus

$$Z = R + L(-a + j\omega) + \frac{1}{(-a + j\omega)C} \tag{60}$$

For any given numerical values[1] of a and ω the impedance may then be found from the above relation. The resonance curve for a damped e.m.f. is shown in Fig. 47. By comparison of this figure with Fig. 45 which is plotted to the same scale it is evident that the resonance of a circuit to the excitation of a damped e.m.f. is much "flatter" than it is for a sustained e.m.f.

This is also evident, qualitatively, by inspection of the impedance $Z = R + Lp + \frac{1}{Cp}$. If $LCp^2 = -1$ then $Z = R + j0$. But for $p = -a + j\omega = -a + j\frac{1}{\sqrt{LC}}$ it is evident that the reactance component of Z will not vanish.

Resonance Curves for Coupled Circuits.—In the case of the circuit of Fig. 43 we are interested in the current in branch 2 for an e.m.f. in branch 1, that is in the driving-driven point impedance. Solving equation (57) gives the value of this impedance as

$$Z_{12} = \frac{1}{M\omega}\left[\frac{(L_1C_1\omega^2 - 1)R_2}{C_1\omega} + \frac{(L_2C_2\omega^2 - 1)R_1}{C_2\omega}\right] - \frac{j}{M\omega}\left[M^2\omega^2 + R_1R_2 - \frac{(L_1C_1\omega^2 - 1)(L_2C_2\omega^2 - 1)}{C_1C_2\omega^2}\right] \tag{61}$$

If the two circuits have different values of LC there will be two maxima to the current as the impressed frequency is varied and the resonance curve will be of the general form shown in Fig. 48.

Transients in Coupled Circuits.—As has been said before, a transient current which is a damped sinusoid accompanies any change in the character of the e.m.f. impressed on a circuit so

[1] The current, $i = v/Z$, where Z is found by equation (60), is the forced current. When the impressed e.m.f., v, has a damping, a, small as compared to that of the natural oscillation of the circuit, the latter dies out in a relatively short time and the forced current only need be considered. This is the case for Fig. 47. When, however, the damping of the impressed e.m.f. approaches that of the natural oscillation the transient must be calculated. For a general solution of the transient see paper of J. R. Carson, Phys. Rev. vol. 10, p. 217, 1917.

that the resultant current in a circuit of one degree of freedom like that of Fig. 39 is the sum of two component currents, one a forced oscillation and the other a natural oscillation. The value of the forced oscillation is found by dividing the e.m.f. by the impedance of the circuit. Thus if the impedance is Z and the e.m.f. is $E\epsilon^{j\omega t}$ this current is $(E\epsilon^{j\omega t})/Z$. The natural oscillation is of the form $I_0\epsilon^{jxt}$ where jx is the complex frequency constant of the circuit.

In case the circuit has two degrees of freedom there will be, in addition to the forced oscillation, two natural oscillations so that the resultant current will be of the form

$$I\epsilon^{j\omega t} + I_0\epsilon^{jxt} + I_0{}'^{jyt}$$

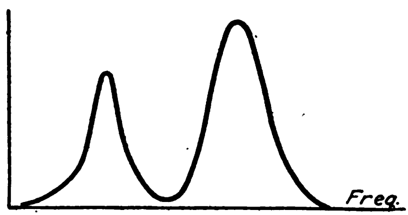


FIG. 48.—Resonance curve for circuits of Figs. 41 and 42.

where jx and jy are the frequency constants determined as in equations (50) and (51).

Frequency or Wave Meter.—If the coupling between two tuned circuits as 1 and 2 of Fig. 43 is very small, then the impedance of circuit 1 and hence the current in it is but little affected by the presence of the coupled circuit 2. It has also been pointed out that for very loose coupling and for a given value of the e.m.f. active in circuit 1 the driving-driven point impedance is made a minimum, and hence the current in circuit 2 is made a maximum, if circuit 2 is tuned to the frequency of the e.m.f. active in circuit 1. Hence if it is desired to determine experimentally the frequency of the e.m.f. active in a circuit, say 1, it is only necessary to couple to it very loosely a second circuit, say 2, the tuning of which may be varied and then to

adjust this tuning until a maximum response is obtained. The frequency of the current and e.m.f. of circuit 1 is then obtained from the constants of circuit 2 by the relation $f = \frac{\omega}{2\pi}$ where $L_2C_2\omega^2 - 1 = 0$.

Circuit 2 is for such purposes usually constructed with a fixed inductance of known value and an adjustable condenser, the value of which in each position is known. To determine when the response is greatest it is necessary to use a detector of some sort with the circuit. The connection may assume any of the several forms described in Chapter IV but Fig. 49 is typical. The circuit with its detector and current-indicating device is then known as a frequency meter or "wave meter"

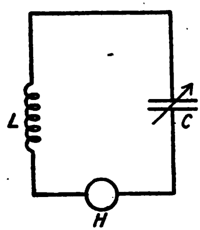


Fig. 49.—Use of tuned circuit as a wave meter.

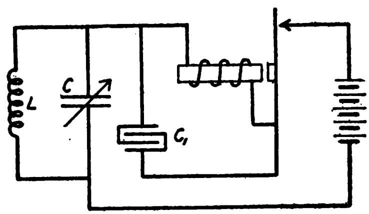


Fig. 50.—Wave meter circuit with "buzzer" excitation.

as it is more commonly called. The term wave meter arises from the fact that, as will be seen on page 112, there is a simple relation between the frequency and the wave length with which periodic electromagnetic disturbances are transmitted through space.

Buzzer Excitation.—If to the wave meter there is added a "buzzer" or interrupter and a battery as shown in Fig. 50 then the circuit may be used to produce damped sinusoidal currents. Such a connection is frequently convenient where it is desired to induce in a circuit currents of a known frequency. In

that case the "buzzer circuit" must be loosely coupled to the desired circuit. A condenser C_1 shunts the windings of the buzzer.

Experimental Determination of Coupling.—If a variable condenser which is calibrated to read capacity is available, then using the buzzer circuit just described, a measure may be made of the inductance of a coil of unknown constants. To accomplish this, the condenser and a current-indicating device are connected to the unknown coil to form a wave-meter circuit as explained above. The wave-meter circuit so formed is then loosely coupled inductively to the buzzer circuit of which the frequency is assumed to be known. The circuit with the unknown inductance L is then tuned to this frequency by varying the capacity. The frequency, $\frac{\omega}{2\pi}$, and capacity C being known, the value of L is found from $\omega^2 = 1/LC$.

If it is desired to find the coupling of two inductances as L_1 and L_2 they may be connected in series without disturbing the relations in space and their total inductance, say L_x, determined as above. Suppose a current i flowing through the two coils in series then the e.m.f. across the first coil is $L_1pi + Mpi$ where Mpi is the e.m.f. due to the mutual inductance M, induced by the current i in the second coil. Similarly, the e.m.f. across the second coil is $L_2pi + Mpi$. The total e.m.f. is then

$$L_1pi + 2Mpi + L_2pi \text{ which must equal } L_xpi.$$

Hence

$$L_x = L_1 + 2M + L_2 \tag{62}$$

Similarly if the connections of one coil are reversed so that the current at any instant flows through it in the reversed direction with reference to the other coil, then the inductance of the combination, say L_y, is given by

$$L_y = L_1 - 2M + L_2 \tag{63}$$

Hence

$$4M = L_x - L_y$$

When the mutual inductance is known and also the various

self-inductances the coupling may be calculated from its defining equation as given on page 77.

Equivalent Circuit of a Transformer.—It is sometimes convenient to replace an inductive coupling such as is given by the transformer of Fig. 43 by an equivalent conductive coupling like that of Fig. 41. To determine such an equivalent consider the two circuits of Fig. 51. Write the equations for the e.m.f.'s in each branch and note that if the circuits are equivalent the equal e.m.f.'s v active in each must produce equal currents. Thus

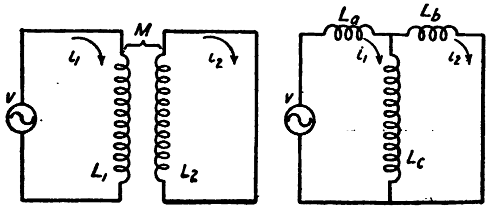


FIG. 51.—Transformer and its equivalent T-circuit.

$$v = L_1 pi_1 - Mpi_2 \qquad v = L_a pi_1 + L_c pi_1 - L_c pi_2$$
$$0 = -Mpi_1 + L_2 pi_2 \qquad 0 = L_b pi_2 + L_c pi_2 - L_c pi_1$$

Hence

$$L_1 = L_a + L_c \tag{64}$$

$$L_2 = L_b + L_c \tag{65}$$

$$M = L_c \tag{66}$$

Spark-gap Excitation.—For the circuit of Fig. 38 it has been shown how the breaking down in resistance of the spark gap G allows an oscillatory discharge of the condenser C, the frequency of which is determined by L, C and R. When this discharge current has damped out, the gap resistance has been restored to its normal value, except as altered by changes in the ionization of the air or gas, occasioned by the previous discharge. With each break of the circuit of the primary of the induction coil there is induced in its secondary and hence applied across the

gap the e.m.f. required to cause the spark. Hence in the antenna circuit there flows a succession of damped sinusoidal currents of the character described above. In ordinary operation the gap resistance returns to its normal value in the interval between these successive damped currents and hence the number of oscillatory sparks per second is controlled by the interrupter in the primary of the induction coil. This latter number is called the group frequency to distinguish it from the radio frequency or oscillating frequency of the antenna.

Periodically recurring spark discharges may, however, be obtained by impressing across the gap the e.m.f. obtained from an ordinary alternator. In that case it is usually necessary

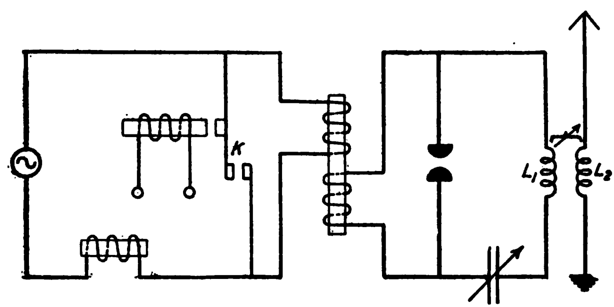


FIG. 52.—Exciting spark circuit by an alternator.

to step up the voltage derived from the alternator by a transformer of the ordinary type used in electrical power engineering, except that there is more than the usual magnetic leakage. It is customary also to use a separate tuned circuit for the spark gap and to couple that circuit with the antenna circuit as shown in Fig. 52. The two coils L_1 and L_2 coupling the circuits are frequently spoken of as an oscillation transformer. A relay key K for controlling the current to the power transformer is also shown in the figure.

In the circuit under discussion the length of the spark gap is adjusted so that only once in each alternation of the e.m.f.

from the generator does the gap break down and the oscillating discharge occur. The group frequency is then twice the frequency of the alternating-current supply. The two frequencies of the natural oscillations in the coupled circuits may be found by the method of page 76.

The frequencies may be determined experimentally by coupling a wave meter to the tuned circuits. When the frequency of the wave meter is approximately that of either of the two oscillations under consideration there will be a peak in the current-frequency curve obtained by it.[1] If the coupling is close, the curve will be of the form shown by the full line in Fig.

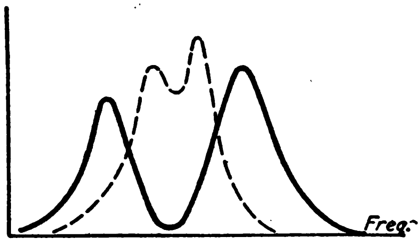


FIG. 53.—Resonance curves for circuit of Fig. 52.

53. If the coupling is loose, the curve will be found to be similar to that of the dotted line in Fig. 53.

Curves showing the currents in the primary, that is, branch 1, and in the secondary or branch 2 are given in Fig. 54.

Quenched-gap Excitation.—If a circuit of but a single degree of freedom is excited there will be of course only one frequency of natural oscillation. If, therefore, in the coupled circuits 1 and 2 of Fig. 52 it is possible to arrange that after excitation has occurred branch 1 is automatically opened so that no current flows in it then except for the instant the current flows in branch 1 there should be developed in branch 2 but a single frequency oscillation. This is accomplished by using for the spark gap

[1] For a further discussion of this matter see Chapter VIII, pages 132–137.

a so-called "quenched gap," that is a gap of small length which offers a high resistance to the current which flows after a spark has passed across it. The current in the primary when using a

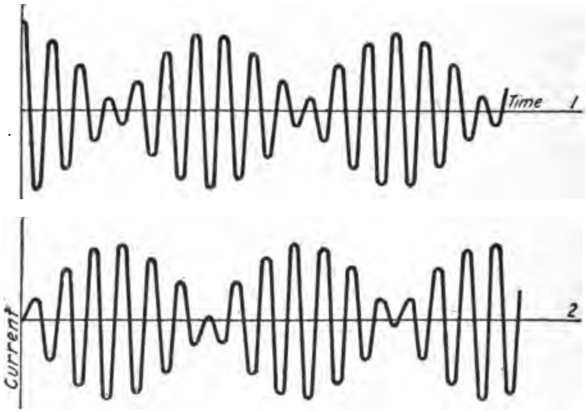


FIG. 54.—Currents in circuits 1 and 2 of Fig. 52

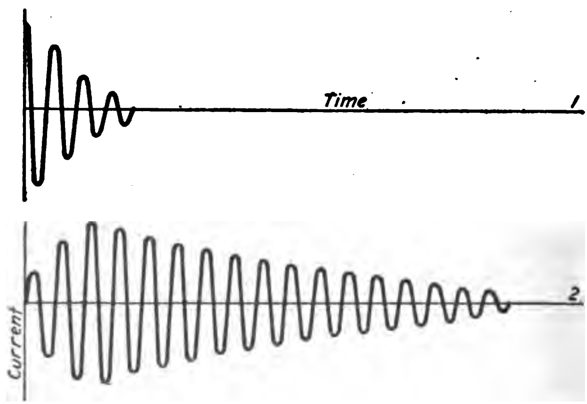


FIG. 55.—Effect of quenching on currents of Fig. 54.

quenched gap is then of the form shown in 1 of Fig. 55, and the resulting current in the secondary is then of the form shown in 2 of the figure.

Synchronous Rotary-gap Excitation.—In the case of the so-called synchronous rotary gap the conditions are similar to those in Fig. 52 except that one terminal of the gap is fixed and the other is made to approach and then recede from the fixed terminal. This motion is so arranged that at the moment in the cycle of the e.m.f. from the generator when the charging current is at its peak the two gap terminals are closest together so that then a heavy spark passes. The arc thus formed is drawn out by the receding electrode and the current damps out. In order to assure synchronism of the motion of the electrode and the e.m.f. wave the rotating electrode is connected to the shaft of the alternating-current generator.

Impulse Excitation.—Those methods of producing natural oscillations of relatively small damping by the actual or equivalent removal of a source of highly damped natural oscillations from a coupled circuit, of which the quenched gap is an example, have been classified by the Standardization Committee of the I.R.E. under the term "Impulse Excitation." Thus "Impulse excitation is a method of producing free alternating current in an excited circuit in which the duration of the exciting current is short compared with the duration of the excited current." "The condition of short duration," as the Committee note and as we have seen above, "implies that there can be no appreciable reaction between the circuits."[1] The definition of impulse excitation is seen to be quite broad.

In mechanics the word "impulse" signifies a discrete blow, such as might be occasioned by the impact of a moving body as distinct from a sustained force. In electrical terminology the word impulse is usually taken to mean an electrical or electromagnetic blow. Thus the sudden impact with a body of a number of electrons would be an electrical analogy. Instances of this character do take place as the student of X-rays well knows. The use of the word impulse in the standardized

[1] "Report of the Committee on Standardization for 1915," Institute of Radio Engineers.

definition is not then in rigorous conformity to its other uses, but is to be taken rather as representing the ideal and limiting case toward which the transmitting systems involving natural oscillations are tending.

Close approximations to the ideal case may, however, be obtained by exciting the secondary circuit through the dis-

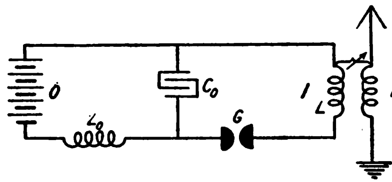


Fig. 56.—Impulse excitation.

charge of a condenser in the primary circuit. The damping of the primary circuit may then be made so high that the condenser discharge is non-oscillatory. Such a system is shown in Fig. 56. A battery *O* supplies through an inductance L_o a charging current to a condenser C_o. The charging current is small but the discharge current when the gap *G* breaks down

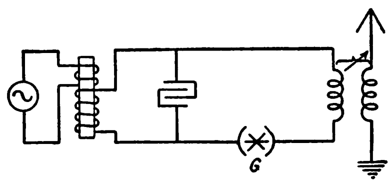


Fig. 57.—Alternator and rotary gap discharger.

may be many times as large. This sudden surge of current in circuit 1 induces in circuit 2 a similar surge which is followed by the oscillations natural to that circuit. The reaction of circuit 1 on circuit 2 may be made of short duration, but persists for an appreciable time relative to the period of the latter which is a high-frequency oscillating circuit. The gap *G* may

be a stationary gap or a rotary gap. In general the gap is a multiple gap made up of a series of short gaps, and in that respect, as in others, is similar to the quenched gap discussed above. The character of the discharge is to some extent controlled by the constants L and C_0 of circuit 1 but largely by the quenching of the gap. This depends upon its design, upon the methods for cooling the electrodes and upon the kind and pressure of the gas between the electrodes.

The group frequency of the discharge is sometimes controlled by the speed of the rotary gap and sometimes by the introduction of a so-called tone circuit, the resonant frequency of which lies within the limits of audibility. A pure tone, that is, a periodic succession of impulses, is sacrificed to the convenience of obtaining higher e.m.f.'s across the condenser which accom-

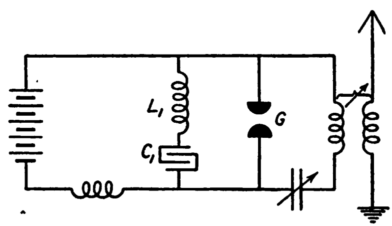


FIG. 58.—Tone circuit.

panies the use of an alternating-current generator and a step-up transformer. In Fig. 57 is shown such a system using an alternating-current source and a rotary gap. In this system it is evident that the nominal group frequency will depend upon the speed of the rotary gap G, but that the alternating-current e.m.f. across the condenser may at times be too low to discharge even though the gap is in position for a discharge.

In Fig. 58 is shown a system involving the "tone circuit" mentioned above. The tone circuit evidently acts in the nature of an alternator, of frequency controlled by its constants L_1 and C_1, which will cause one spark (at least) for each half cycle of the oscillatory current with which it discharges across the

gap. The system of Fig. 58, except as to differences introduced by the gap, is similar to the system for spark excitation described on page 88.

Summary.—All systems for the production of high-frequency e.m.f.'s by the natural oscillations of an electrical circuit containing both types of storage reservoirs are based upon fundamental principles which have been developed earlier in the chapter. Thus it has been shown that there will be present in any network a forced current and a transient current. This transient may be resolved into a number of components equal to the number of degrees of freedom of the electrical system, the frequency constants of each component being a function of all the resistances, capacities, and self and mutual inductances of the entire network. The production of a single damped sinusoid becomes then a matter of effectively reducing the circuit to a single degree of freedom and of minimizing the duration of the forced current. As long as the network has more than one degree of freedom there must be a component in the transient current for each degree of freedom and hence the desired single damped sinusoid can be obtained only during that portion of time in which the number of degrees of freedom has been reduced to one by opening the circuit at the proper point.

CHAPTER VI

PRODUCTION OF UNDAMPED HIGH-FREQUENCY CURRENTS

In the preceding chapter we have seen how damped oscillations of any desired high frequency may be obtained by utilizing the phenomenon of transient or natural oscillations. The principles underlying this general method have been stated and some illustrations given. It is intended in this chapter therefore to discuss methods for producing continuous, that is, undamped, high-frequency currents.

Inductor Alternators.—The most obvious method is to use an alternator similar in principle to the alternators used for generating frequencies such as are utilized for electric light and power but designed mechanically and electrically for high frequencies. Alternators may be of three types, namely: (*a*) fixed magnet poles and movable armature; (*b*) fixed armature and movable magnet poles; and (*c*) fixed magnet poles and fixed armature coils and movable inductors. These latter are strips of magnetic material which are moved in and out of the fixed magnetic field formed by the direct current in the pole windings so as alternately to increase and decrease the flux linking with the armature coils. The difficulties incident to rotating coils, whether as armatures or as field windings, to which connections for a current flow must be made, are therefore avoided. The most successful alternators for high frequency are therefore of the "inductor" type. Such machines are generally known by the name of the designer, as for example, the Alexanderson alternator.

Alexanderson Alternator.—In Fig. 59 is shown a partial section of the Alexanderson alternator. The shaded portions

N and S are pole teeth. The magnetic flux is provided by two coils A and A which are concentric with the axis. The poles of the magnetic field on one side of the rotor R are thus all north and those on the opposite side all south. The armature windings appear in this sectional view as small circles above and below the laminated pole pieces N and S. The rotor is a steel disc in which slots have been milled leaving thin strips of steel between the slots. The latter are filled with a non-magnetic material. In the figure the cross-section is taken through one of these slots.

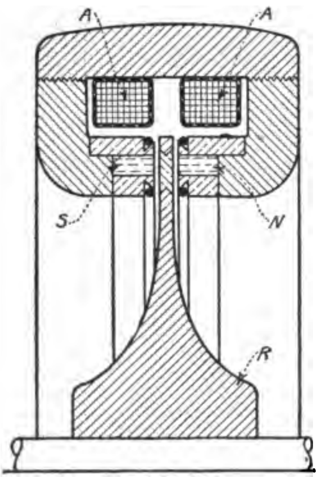


FIG. 59.—Partial cross-section of inductance alternator.

As the rotor turns non-magnetic slots and magnetic spokes alternately pass through the air gap between the teeth N and S. The magnetic reluctance and hence the flux across each gap is thus successively varied from a maximum to a minimum. In Fig. 60 is shown schematically a part of the rotor and of the armature winding on the other side of it. In the position shown it is evident that in the loop formed by conductors 5 and 6 the flux passing through spoke d is active. As the rotor turns this flux will be reduced by the motion into this space of the next slot. As regards the loop formed by conductors 4 and 7 the flux passing through spokes c, d, and e is active at the moment shown. As the rotor turns through the angle subtended by one spoke this flux will be reduced by the passage out of the loop of one spoke, *e.g.*, spoke c. Hence the rotation of the disc through the angle corresponding to one spoke increases or decreases the flux through each of the loops 2–5, 3–4, 4–7, 5–6, etc., by the same amount. This unusual winding, developed by the inventor, permits therefore the use of only two-thirds as many slots for

the armature windings as there are effective number of poles. In this way the frequency generated may be increased without increasing either the speed of rotation or the number of poles. The frequency is obviously the product of the number of teeth (on one side of the rotor) and the revolutions per second of the rotor. Frequencies of 200,000 cycles have been obtained with the generator described.

Goldschmidt Generator.—A production of high frequency by utilizing the phenomenon of a rotating magnetic field, as exemplified in the induction motor of power engineering, and the phenomenon of tuned circuits is accomplished in the Goldschmidt alternator.

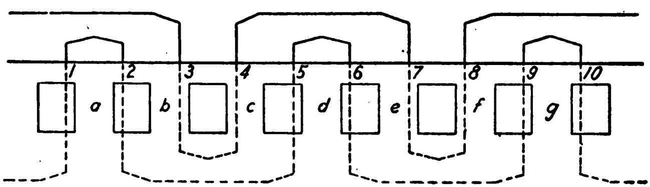


FIG. 60.—Armature winding in Alexanderson alternator.

The induction motor consists of a "stator" and a "rotor." The stator is a group of fixed windings so arranged that alternating currents in them produce a resultant magnetic field the direction of which is changing, that is rotating, at a speed determined by the frequency of the alternating-current input. The "rotor" is a short-circuited armature in which the changing flux induces a current. The reaction of this current and the field results in a rotation, the armature being effectively dragged around by the rotating field. This is, of course, because the induced currents are such as to oppose the cause (that is the rotation of the field). Such an action requires polyphase currents or what amounts to the same thing currents differing in phase. A single-phase alternating current flowing in a coil does not, of course, produce a rotating magnetic field but a field alternating in direction and sinusoidal in its intensity. A rotor placed in

such a field would not rotate. It would have induced in it, of course, an alternating current tending to oppose the magnetic field just mentioned.

A sinusoidal magnetic field may, however, be considered to be the resultant of two equal and opposite rotating fields. That is, it may be represented by two conjugate vectors, each of half the maximum value of the field intensity. If a rotor is driven or rotated at the same speed as one of these component rotating fields, this component will induce no current in it because there is never any change in the flux through the rotor

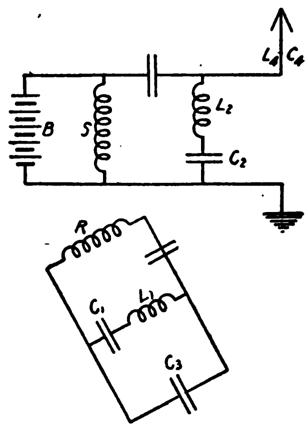


FIG. 61.—Circuits of Goldschmidt generator.

due to that component. The relative rotation of the rotor and the other component of the magnetic field will, however, be double the space rotation of the rotor because field and rotor are moving in opposite directions. If the angular velocity of the rotor in space is ω (where ω is the angular velocity of the vector representing the alternating current in the stator), then there is induced in the rotor an alternating current of twice the frequency of that flowing in the stator.

This general principle is applied in the Goldschmidt alternator as illustrated in Fig. 61. The windings of the stator and of the

rotor are, of course, distributed in slots but for convenience of illustration they are represented by coils in the figure. A stator S is supplied with direct current from a battery B. The rotor R is driven with an angular velocity ω. The current induced in the rotor is then of frequency f, equal to $\omega/2\pi$. The rotor circuit is made of low impedance to that frequency by tuning the circuit of L_1 and C_1. The field caused by this current is alternating with reference to the rotor. But the rotor is revolving, hence if this field is replaced by two components oppositely rotating with reference to the rotor it is seen that one component will be fixed in direction with reference to the stator while the other will have a velocity of rotation relative to the stator of 2ω. This latter component will induce a current in the stator of frequency $2f$.

The stator is tuned to this double frequency by L_2 and C_2. With reference to the stator this current produces an alternating field which may be resolved into two components rotating in opposite directions with velocities of 2ω.

Remembering that the rotor is revolving with a velocity ω it will be seen that with reference to the rotor these velocities become ω and 3ω. The rotor is also tuned for the frequency $3f$ by C_3. The current of this frequency may be seen to give rise in the stator to currents of frequencies $2f$ and $4f$. For the latter frequency L_4 and C_4 of the antenna circuit may be tuned, resulting in a frequency of $4f$ in the antenna.

Frequency Changers.—The Goldschmidt generator discussed above is an illustration of both a generator and a frequency changer, since it first generates a frequency of f and then by so-called "reflections" converts that current into a higher frequency.

Of the various other methods which have been proposed for increasing the frequency of the current generated by an alternator two are of practical interest. One method involves the use of iron-cored transformers similar to those of power engineering. Although several different combinations have been proposed the following is typical. Two transformers are used as

1 and 2 of Fig. 62. The primaries each have two windings. Through one primary winding of each transformer passes sufficient direct current to bring the induction B of each transformer up to the knee of the magnetization, or flux-ampere-turns curve, as shown in Fig. 63. To the other winding is supplied the alternating current which is to be changed. The polarity of the transformer windings is so arranged that in transformer 1 the alternating current is increasing the flux at the same time that in transformer 2 it is decreasing it. In transformer 1 then there is in this half cycle but a small increase in flux, while in transformer 2 there is a large decrease. Conversely for the next half cycle, the flux in 1 decreases, while that in 2 makes but

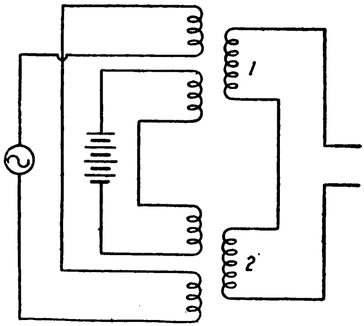


FIG. 62.—Transformers connected to give double frequency.

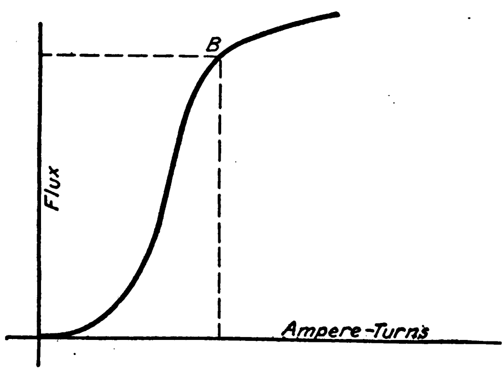


FIG 63.—Magnetization curve.

a small increase. The e.m.f.'s developed in the secondary windings of the two transformers are then of the general form shown

in Fig. 64 (1) and (2). The secondaries are connected in series and there is then active in the circuit thus formed an e.m.f. as given in Fig. 64 (3). Of course, if the secondary circuit is tuned to the double-frequency current thus produced, this current will be more nearly a pure sinusoid.

The analogy that may be drawn between the curves of Figs. 20 and 25 of Chapter III and that of Fig. 63 may be of interest to the student.

The other method which was mentioned above involves the use of some device giving an output current with a component proportional to the square of the input current as an electrolytic cell or a vacuum tube. The mercury-arc rectifier has also been used for this purpose.

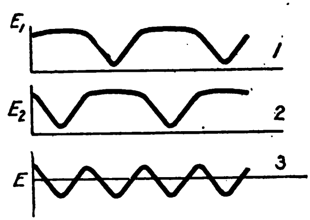


Fig. 64.—E.m.f's in circuit of Fig. 62.

As to these two methods of producing higher frequencies, it is evident that the one involving iron-core transformers will involve high losses due both to hysteresis in the iron and to eddy currents. Large outputs by this method have, however, been obtained. As to the second method, it is evident that an alternator is required in addition to the frequency changer. For long-distance transmission as will be seen in Chapter VII frequencies in the neighborhood of 30,000 cycles are used. These are well within the range of the Goldschmidt or Alexanderson generators. For short distances, however, where higher frequencies are used, the power requirements are less and the vacuum tube furnishes a convenient generator.

Vacuum-tube Oscillator.—Any device which repeats with amplification may be used as a generator of oscillations when connected into the proper circuit. For example, a telephone receiver placed in front of an ordinary telephone transmitter forms a repeating system. If such a combination is connected in series there results in general a "singing" circuit. The fre-

quency of the oscillations is in this case determined by the natural frequency of the mechanical vibrating system. Such a "howler set" is shown in diagram in Fig. 65.

In case the repeating device does not itself have a natural frequency from its mechanical structure, as is the case with the vacuum tube, the frequency of oscillation may be controlled entirely by the electrical constants of the circuit. Either the output or the input circuit may contain a tuned or resonant circuit. The output must, of course, be coupled to the input. In Fig. 66 there is shown such a connection, the output circuit O being coupled inductively to the tuned input circuit I.

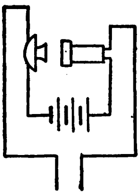


FIG. 65.—"Howler set"

The operation is then as follows: Imagine some slight variation in the output current. This will excite in the input circuit natural oscillations. These are impressed on the grid and hence give rise to similar but amplified oscillations in the output circuit. By the inductive coupling these excite forced oscillations in the tuned input circuit. These in turn are repeated in the output and the alternating current builds up to a steady value which is determined by the $E_C I_B$ characteristic of the tube. In Fig. 67 is shown a typical characteristic. Below it is the e.m.f.-time curve for the increasing voltage impressed on the grid; on the right-hand side of the figure is shown the current-time curve. The average value of the current is the dotted line. Why the current increases to a definite steady value is then evident. The alternating current thus produced is obviously not a pure sinusoid but contains higher

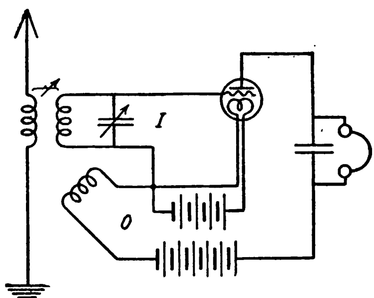


FIG. 66.—Oscillating vacuum tube.

harmonics. The current as read on a direct-current ammeter in the plate circuit will be as indicated by the dotted line.

Suppose the tube to receive an e.m.f. from the antenna as well. It is then in condition for heterodyne receiving as described in connection with Fig. 35 of page 61. This connection of the tube is called a "feed back" and the tube is said to be

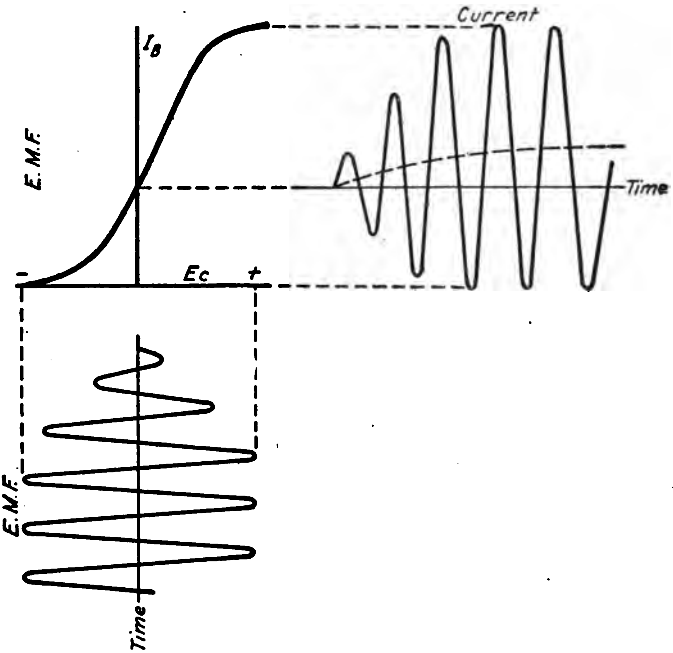


FIG. 67.—Simplified curves of input and output for oscillating tube.

"oscillating." In operation the input circuit must be slightly off the tune of the signals to be received in order to give an audible "beat note."

The oscillations in the output O are, of course, impressed on the antenna circuit. The same scheme of connections as in Fig. 66 may, therefore, be used to excite continuous oscillations in the

antenna. That is, the oscillating vacuum tube may be used either as a heterodyne receiver of continuous waves or as a generator of them. When in the latter use interruptions in the continuous wave may be caused by inserting a key so as to open either the input or the output. Or such interruptions might be caused by having the up position of the key alter either the L or C of the input circuit while the down position restores the tuned circuit to the desired frequency. If the change in the L or C is large the alternating current caused by the oscillating tube when the key is up will differ so much from the resonant frequency of the antenna circuit that but a negligible current will flow in the antenna.

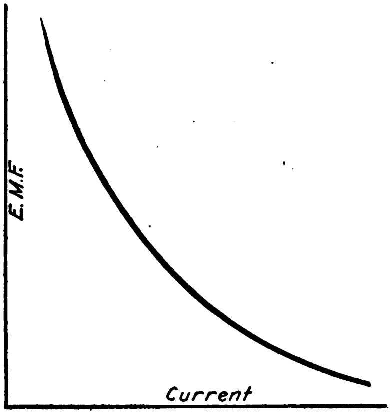


FIG. 68.—Static characteristic of an arc.

Arc Generators.—The Poulsen arc generator which is very generally in use depends for its operation on the phenomena of the discharge of electricity through gases which have been outlined in Chapter III.

Let an arc be struck between two metal electrodes by bringing them into contact and then drawn out by separating the electrodes. If the e.m.f.-current characteristic of the arc path is plotted it will be of the form shown in Fig. 68. Since this curve is obtained by impressing steady direct e.m.f.'s it is called the "static characteristic." This idea but not this name has been met before; thus in Fig. 67 the I_B-E_C curve is the "static characteristic" of the tube. From the other plots of the same figure a "dynamic characteristic" or I_B-E_C curve could be obtained.

The arc is instable, for if an increase of current is produced by increasing the e.m.f. applied to the arc path there results a

greater heat at the cathode and hence greater ionization. The greater number of electrons thus made available for conduction results in a still greater conductivity, that is, a lower resistance. The impressed e.m.f. is therefore enabled to cause still more current and this increase of current continues, the voltage required to maintain the arc decreasing toward a definite minimum as indicated in Fig. 68. These higher currents result in a rapid disintegration of the material of the electrodes and other effects that need not further concern us. The arc is said to have a "falling characteristic."

It is because of this falling characteristic and the consequent instability that arc lamps supplied from constant-voltage mains must have ballast resistances in series with them. For otherwise, suppose the voltage impressed on the arc was just that required to maintain the current, then any momentary increase in current would lead to the sequence of events just described. On the other hand, a momentary decrease in current would reduce the heating of the electrodes (and gas path) and hence require a higher voltage across the arc. But a constant e.m.f. would not adapt itself to this condition, and hence the current would decrease rapidly. The required e.m.f. increasing and being always above the value furnished by the constant voltage mains, the arc would go out. To avoid this, sufficient resistance must be inserted in the circuit so that for any momentary decrease in current the decrease in e.m.f. required across the resistance and hence the increase in e.m.f. available at the terminals of the arc will be sufficient to increase the arc current to its former value.

If the e.m.f. impressed on the arc circuit is not constant but is alternating, then it is evident from a consideration of the phenomena of ionization that the relation of current to voltage will not be the same for an increasing voltage as for a decreasing voltage. Thus it will be recognized that when the current is decreasing there will be present in the arc stream a larger number of "carriers" than there were when the current had the same

value but was increasing. The e.m.f. required is then less for the case of the decreasing current. The plot of e.m.f. across the arc against current through the arc will then be of the form shown in Fig. 69.

When the sinusoidal e.m.f. is impressed, the electrodes are cool and but a small current flows until the e.m.f. has attained the value corresponding to point *a* on the curve. Thereafter the current rapidly increases to a maximum, the required e.m.f. decreasing (see point *b*). The electrodes are now hot and the current may be maintained by a small voltage as pointed out above. When the e.m.f. reverses its direction, the same phenomena occur, giving rise to the lower half of the curve.

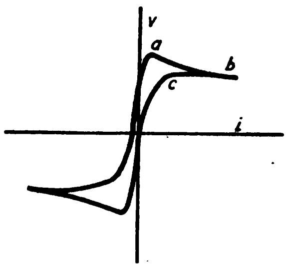


FIG. 69.—Dynamic characteristic of an alternating-current arc.

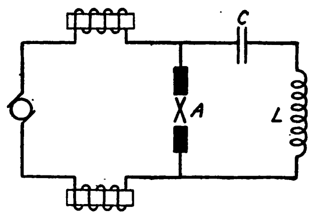


FIG. 70.—Direct-current arc with tuned circuit.

Consider now the circuit of Fig. 70 in which is shown an arc supplied through large inductances by a direct-current generator. This circuit is similar in form to some of those involving spark gaps which were discussed in Chapter V. Assume for the moment that the separation of the arc electrodes is greater than the distance the generator voltage can break down so that it is not yet a conducting path. Assume also that the generator has but recently been connected to the circuit so that the charging current to the condenser is still flowing. At this moment let the arc electrodes be brought near enough for the arc to form. There passes through the arc path then not only the discharge current from the condenser but also the current from the gen-

erator which the series inductances tend to maintain constant. The discharge current from the condenser flowing in the tuned circuit *LCA* operates to charge the condenser in the opposite direction. During this time the arc path is carrying the steady current delivered by the generator. When the condenser discharges again, its polarity is such that its current through the arc is of course opposite to the current from the generator. The net current in the arc path may then be reduced to zero, in case this discharge current is at least as large as the steady current from the generator. If this is so, the arc goes out. This leaves the condenser *C* unable to complete the reversed discharge which we are discussing, and aiding in e.m.f. the generator. The steady current from the generator is of course diverted from the arc path and is available for reducing the condenser charge to zero and then for charging it again to the e.m.f. at which the arc path broke down. When this breakdown e.m.f. is reached the cycle repeats. The period of the alternations thus produced is controlled only in part by the tuning of the condenser circuit since the time between the reduction of the arc current to zero and the next discharge is dependent upon the rapidity with which the generator can recharge the condenser. The two parts of the cycle, namely, discharge and recharge, will obviously occupy different times, the former being the shorter and also directly dependent on the tuning of the condenser circuit. The wave form produced will not be a pure sinusoid but will consist of a fundamental and several higher frequencies or harmonics as they are called.

A modification of this cycle is possible in case, immediately following the reduction to zero of the current through the arc path, a value of e.m.f. is reached in the reversed direction sufficient, in the highly ionized condition of the arc path, to start an arc in that direction. In this case there occurs an oscillating discharge through the gap until the damping of the circuit *LCA* reduces the voltage too far for further discharges. The generator then recharges the condenser and the damped oscillating dis-

charge will be repeated. In this case the action of the circuit is that of the spark-gap circuits described in Chapter V and is not strictly speaking a normal arc phenomenon.

The possible types of oscillations in an arc circuit are for convenience commonly referred to by numbers. Thus, type I is for the case where the current from the condenser circuit is always less than that from the generator. This type was not discussed above as it is of small practical importance. It is characterized by the fact that the current through the arc is always greater than zero. Type II which was the first to be discussed is characterized by the fact that the maximum current in the condenser circuit is just equal to the supply current from the generator so that the arc conducts in one direction only. Type III is similar to type II except that the arc reverses.

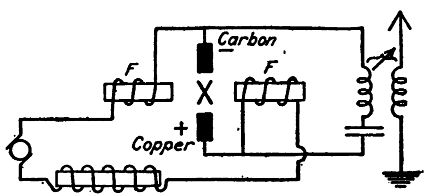


Fig. 71.—Poulsen arc.

The Poulsen arc makes use of oscillations of type II. The essentials of the circuit are shown in Fig. 71. The electrodes are carbon (−) and copper (+), the latter being cooled by running water. A magnetic field for blowing out the arc is provided by the coils *F*, *F*.

In the arc discharge as described above, it is evident that a reduction in the time required to recharge the condenser is obtained if the voltage across the arc does not reverse. The voltage at which the arc ignites is determined by the separation of its electrodes, by their temperature and by the ionization condition of the intervening gas. Following ignition the voltage across the arc rapidly falls as the current increases so that the greater part

of the current passes at a low voltage as indicated by the nearly horizontal portion of Fig. 68. As the current through the arc decreases due, as explained above, to the reversed discharge of the condenser, a value is reached where, as seen in Fig. 68, the voltage across the arc will rise. This phenomenon was described on page 105 in discussing the instability of an arc. This voltage has been called the "extinction voltage" corresponding to the "ignition voltage."

In the Poulsen arc under efficient operation it has been shown[1] that the extinction voltage rises to a value but little less than that of the ignition voltage. If the separation of the electrodes is

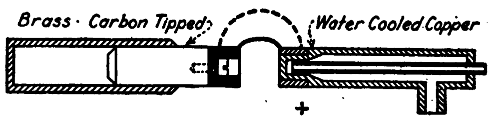


FIG. 72.—Arc paths and electrodes of Poulsen arc.

made too large the reversal of voltage across the arc occurs. If the separation is too small the arc tends to burn as a non-oscillatory arc.

The effect of the magnetic blow out is to blow out the arc, which action is accompanied by a motion of the "craters" or points of contact of the arc stream with the electrodes. The craters move back along the electrodes. The succeeding arc starts in general from the hot edges of the electrodes. Thus in Fig. 72 is shown the position of the arc at ignition and by the dotted line at extinction. The figure also shows the water-cooled anode and the carbon-tipped cathode, both in cross-section. The intensity of this magnetic field is an important factor in the design and operation of the Poulsen arc as has been pointed out by Pedersen in the reference cited. If the field is too strong the arc "craters" travel rapidly into the cooler regions away from the edges of the electrodes, the current rapidly decreases and the extinction voltage rises. This voltage across the arc may then

[1] P. O. PEDERSEN: "On the Poulsen Arc and its Theory," *Proc.* I. R. E., vol. 5, pp. 255–316, 1917.

ignite a new arc across the shorter path between the hot edges of the electrodes. The arcs sketched in Fig. 72 may thus exist simultaneously. The constancy of the frequency generated by the system will thus be impaired.

If the field is too weak Pedersen has shown that the arc will not be completely blown out during the period determined by the tuned circuit. The next arc then occurs between the points on the electrodes to which the previous arc has extended. During the burning this new arc is moved along another "step" by the action of the field. Finally then one of the arcs reaches the extreme possible length and then ignition occurs again at the edges of the electrodes and the cycle repeats. For such an adjustment of the magnetic field the voltage supplied by the direct-current source must be larger than for the case of the proper field intensity because during the passage of the current a larger voltage must be maintained across the terminals of the longer arcs. The conditions for successive arcs are not constant. Hence in this case also there are variations in the generated frequency.

The proper value of magnetic field to use in any case is one which will just blow out the arc during one period, allowing a new arc to form at the beginning of the next period. The most suitable field intensity will therefore depend upon the frequency it is desired to generate. The relations developed by Pedersen show this field intensity to be approximately proportional to the frequency.

Under the proper conditions of operation the effective value of the radio-frequency current generated will be $\frac{1}{\sqrt{2}}$ times the supply current from the direct-current generator.

CHAPTER VII

RADIO TELEGRAPHY AND TELEPHONY

The Ether.—The medium through which electrical and magnetic forces act is called "ether." With the theories as to its composition or construction we are not here concerned. It is, however, by virtue of this medium which fills all space that electromagnetic disturbances, whether discrete or periodic, are made manifest at a distance. Of the periodic disturbances thus transmitted light and heat are two classes. The periodic disturbances made use of in wireless telegraphy form a third class. The velocity of transmission of all three is the same, namely, 186,000 miles per second or 300,000,000 meters per second.

Wave Motion.—Consider a source of periodic electromagnetic disturbances, as for example, an antenna circuit. Let the disturbance occur every T seconds, that is, have a frequency of $f = 1/T$. Let V represent the velocity with which an electromagnetic disturbance is propagated through the ether. Consider time to begin at the moment when the first disturbance occurs at the source. Then T seconds later the effect of this first disturbance can be detected VT meters from the source. At this instant a second disturbance is occurring at the source. At a time $2T$ the first disturbance will have reached all points distant $2VT$ from the source, the second disturbance will have reached all points distant VT from the source and a third disturbance will be occurring at the source.

Now imagine a series of disturbances occurring at the source of such a nature that their respective magnitudes are proportional to the sine of an angle which increases with the time. At any time t the disturbance at the source may be represented

by $A \sin \omega t$. To all these sinusoidal values the reasoning above may be applied. Thus at the instant when $\omega t = \omega(2T)$, that is $\omega t = 4\pi$, the disturbance occurring at $\omega t = 0$ has travelled to points distant $2VT$ from the source and that occurring when ωt was 2π to points distant VT. Similarly when $\omega t = 4\pi + \theta$ the disturbance which occurred at the time $\omega t = 2\pi + \theta$ is at points distant VT and that which occurred at $\omega t = \theta$ is at points distant $2VT$. That is, there occur at points distant from the source by VT, $2VT$, $3VT$ and so on, disturbances which are exactly in phase with the disturbance occurring at the source. We may then write the disturbance occurring at the

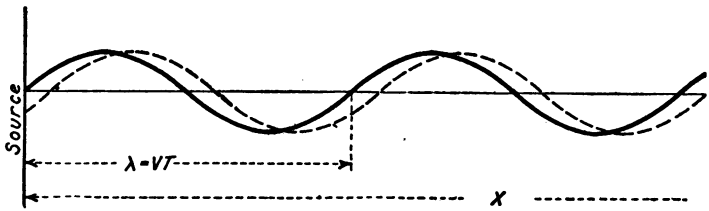


FIG. 73.

point distant $2VT$ as $K_2A \sin (\omega t - 4\pi)$ and that occurring at the point distant VT as $K_1A \sin (\omega t - 2\pi)$ corresponding to a disturbance of $A \sin \omega t$ at the source. The factors K_1 and K_2 are to take into account any reduction in the intensity of the disturbance occasioned by its propagation to a distance.

It is now desired to find what the disturbance is which is occurring at a point distant X from the source. Let λ represent the distance VT for convenience. It takes X/V seconds for the disturbance to travel to the point in question. As above we see that if X is 2λ then the disturbance is $KA \sin (\omega t - 4\pi)$, neglecting for the moment the variations of K with the distance. If X is λ, it is $KA \sin (\omega t - 2\pi)$. By proportion then the disturbance at X will be $KA \sin (\omega t - 2\pi X/\lambda)$. Now T is $1/f$, hence $\lambda = VT = V/f$ and $2\pi X/\lambda = 2\pi f X/V = \omega X/V$. Hence also the disturbance at a point distant X from the source

may be written $KA \sin \omega(t - X/V)$. The disturbance to be expected at any point at some definite time is then a sinusoidal function of the distance to the point; that is, at any instant the disturbance in the ether is proportional to the sine of an angle which is constantly increasing with the distance. Two points which are separated by a distance λ have disturbances of the same character, that is, phase and also magnitude (except in so far as the latter is influenced by the factor K which will be discussed later). The distance λ is called the wave length.

At any instant, say t_1, the intensities of the disturbances at the various points along a line from the source are as shown in the full curve of Fig. 73. At some later instant, as t_2, they will be as the dotted line of that figure.

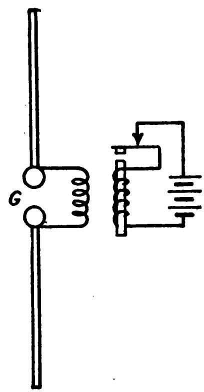


FIG. 74.—Hertzian oscillator.

Oscillator—Simple.—The simplest form of oscillator by which periodic electromagnetic disturbances may be propagated is due to Hertz. It consists as shown in Fig. 74 of a straight wire or rod broken by the introduction of a spark gap G which is supplied by an induction coil. As the e.m.f. from the coil rises, the two halves of the rod become oppositely charged. The two parts of the oscillator form a condenser and an electrical field exists between them. The direction of this field which exists in the space surrounding the oscillator varies from point to point. At all points, however, in a plane perpendicular to the rod and passing through the center of the gap the direction of the field will be parallel to that of the rod.

The act of charging the oscillator means a repulsion of electrons from one gap electrode along its attached rod and an attraction of electrons from the other rod to its electrode. The operation of charging, therefore, is equivalent to a unidirectional flow of current from the extremity of one rod to the extremity of the

8

other. It is evident, however, that this charging current will be zero at the ends and a maximum at the gap. The magnetic field due to this current will form circles concentric with the oscillator. In the plane perpendicular to the oscillator at its center the magnetic field will then always be perpendicular to the electrical field previously described.

Limit the consideration for the moment to effects in this plane. It is evident that at any points to which the electrical and magnetic fields arising from the current in the oscillator may have extended, these fields are at right angles to each other and also at right angles to the radial line connecting the point and the center of the oscillator. In other words, the direction

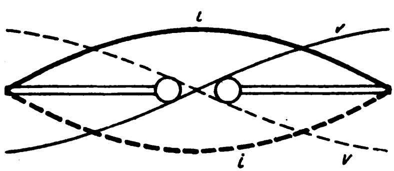


FIG. 75.—E.m.f. and current distribution in Hertzian oscillator.

of propagation of the electromagnetic disturbance is normal to the plane of the electrical and magnetic forces.

Returning to the oscillator, we see that at any instant the plot giving the charging current at each point is of the form shown by the full heavy line of Fig. 75. At some later instant the gap breaks down and an oscillating discharge occurs. At some still later instant the distribution of the discharge current will be similar but opposite to that of the charging current, and as shown by the heavy dotted line.

The counter e.m.f. of the oscillator considered as a condenser, that is the electrical potential of each point, will be as shown by the light lines of Fig. 75. That the potential is a maximum at the extremities of the oscillator is evident from the following considerations. The work required to move an electron to an extremity of the oscillator, hence the work per unit charge, that

is the potential, is a maximum because it must be moved against the repulsions of all the other electrons and for the greatest distance.

Following the breakdown of the gap the potential distribution will be of the form shown in Fig. 75 by the light dotted line. The plot of current and potential do not represent maximum values since obviously the maxima of voltage and current do not occur at the same instant. When the charging current ceases to flow the potential is a maximum. *Vice versa,* when the discharge current is a maximum the condenser is completely discharged, the potential zero and the current in the direction for reversing the charge.

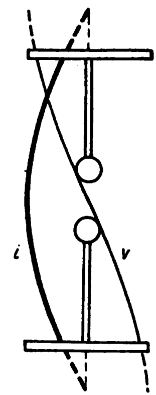


FIG. 76.—Oscillator with lumped capacity.

The point where the potential is always zero is called a node and conversely where it undergoes the maximum variations is called an anti-node. The frequency with which these variations occur is of course determined by the relation of the inductance and capacity, given by equation 42.

Oscillator—Complex.—If the simple oscillator of Fig. 75 is shortened and the equivalent capacity obtained by increasing the capacity by adding wires or plates, there results an oscillator of the more complex form shown in Fig. 76. The maximum current and potential are shown in distribution by the heavy and the light lines respectively. If the lower half is replaced by a conducting plane, the oscillator becomes the Marconi form shown on page 70.

Antenna Design.—An antenna therefore is an oscillator derived from the Hertzian oscillator by using the earth for one-half and by increasing the current in the vertical by increasing the capacity of the extremity of the upper half. This increase of capacity admits of a greater current than can be obtained for the same height in a simple vertical wire.

Of the various forms of antennæ which have been tried at various times but two types are in common use today. These

are the umbrella type and the flat-topped. The umbrella type illustrated in Fig. 77 consists of a vertical conductor from the top of which other conductors slope downward like the ribs of an umbrella. The flat-topped form is usually that of a T although it is sometimes an inverted L. In some cases the horizontal portion is a triangle formed by three flat-topped structures to only one of which vertical conductors are connected. This form is used by the U. S. Navy as illustrated in Fig. 78.[1] The T form is usual on ships. The inverted L with a long horizontal portion is preferred by the Marconi Company.

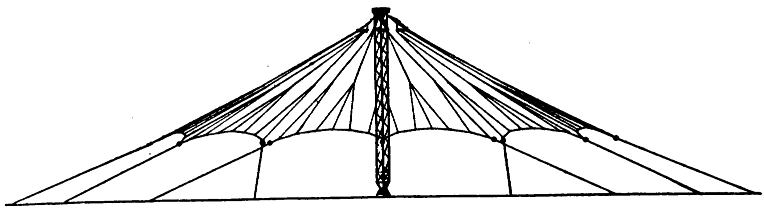


FIG. 77.—Umbrella antenna.

Electromagnetic Radiations from an Antenna.—From a complex oscillator of the type known as an antenna there is propagated by an alternating current in the oscillator an electromagnetic disturbance. Upon the assumption that the earth's surface is flat and a perfect conductor, it may be shown that this disturbance at any point is the same as would be produced by a complete symmetrical oscillator of which the upper half has the form of the actual antenna. Upon this assumption expressions may be written for the intensity of the electrical and magnetic forces at any point. A correction factor allowing for the curvature of the earth may also be introduced. Upon these assumptions the maximum amplitude of the disturbance at a point on the earth's surface distant X from the oscillator is to be found by multiplying its value for $X = 0$ by

$$\frac{1}{\lambda X}\sqrt{\frac{\theta}{\sin\theta}}\,\epsilon^{-\frac{0.0019X}{\sqrt[3]{\lambda}}}$$

[1] E.g. at Arlington, c. f. CAPT. BULLARD: I.R.E., v. 4, pp. 421–47, 1916.

where θ is the angle the distance X subtends at the center of the earth.

The practical conditions, are, however, not so simple. The constitution and electronic behavior under the influence of the

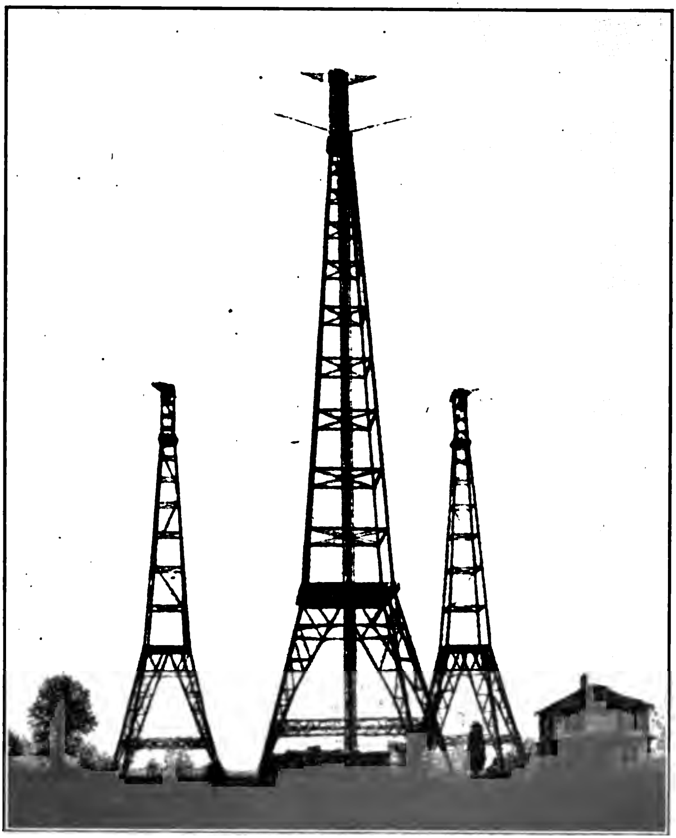


Fig. 78.—Triangular flat-topped antenna.

sun's rays of the upper strata of the earth's atmosphere are as yet undetermined. The surface of the earth is far from spherical, and the local irregularities in curvature due to mountains are

appreciable. In addition, the conductivity of the earth's surface and of its strata vary with wide limits. The atmospheric conditions at various points along the path of a wave will also vary. The result is that the relations obtainable theoretically for the ideal case have not as yet been extended to cover practical cases. Before this can be done more exact data must be obtained by careful measurement.

Attenuation.—While no satisfactory formula of general application exists for computing the attenuation, or the fractional reduction in intensity, of a periodic disturbance as the distance from the source is increased, an empirical formula has been obtained by Austin[1] for one special case. It covers transmission during daylight over sea water between ordinary ships' antennæ. It indicates for K in the wave-motion formula of page 113 a value of

$$\frac{1}{X\lambda}\epsilon^{-\frac{0.0015X}{\sqrt{\lambda}}}$$

where X and λ are measured in kilometers.

The work of Austin also gives for the same conditions a measure of the effective value of A, in the wave-motion formula, provided flat-topped antennæ of heights h_1 and h_2 in kilometers, are used for transmitting and receiving, respectively. The wave-motion equation then gives the disturbance in terms of the effective current in an antenna of about 35 ohms effective resistance. Under the conditions, the value of A is

$$A = 120\pi\frac{h_1h_2}{R}I$$

where I is the effective current at the antinode of the transmitter and R is the resistance of the receiving antenna.

Radiation Resistance.—In the case of the telephone receiver as discussed in Chapter II we found that the impedance was composed of two terms one of which represented the motional im-

[1] L. W. Austin: *Bull.* Bureau Standards, vol. 7, p. 315, 1911; and vol. 11, p. 69, 1914.

pedance. In the case of a transmitting, that is radiating antenna, there is in addition to the impedance of the circuit itself a radiation impedance or rather resistance since it is non-reactive. That is, if an artificial antenna circuit is formed, composed of capacity, inductance and resistance equal to that of the actual antenna, then for the same current the power losses in the case of the artificial antenna will be smaller than for the actual antenna by an amount I^2R_a which represents the power radiated by a current in the antenna of effective value I. Conversely, when an antenna is used for receiving there is an absorption of energy from the surrounding ether which depends upon the electrical field intensity at the antenna.

The radiation resistance is given approximately by the formula $R_a = K\left(\frac{h}{\lambda}\right)^2$ where K is $40\ (2\pi)^2 = 1580$ and the height h and the wave length λ are measured in the same units of length.

In order that the radiation resistance shall be a large part of the total resistance it is necessary that the effective resistance of the antenna and the ground resistance shall both be small.

Ground Systems.—To obtain a low resistance for the part of the earth that serves as the lower plate of the condenser of which the antenna wires and more particularly the horizontal wires form the upper plate, two methods are followed. In one case called the "conductive ground" a network of wires is embedded in the ground, for the entire area covered by the horizontal aerial, and connected to the bottom of the vertical aerial. In the other method of "capacity ground" or "counterpoise" the network of wires is supported above the ground by insulators. The two cases are not very different, particularly when the upper layer of the earth is a poor conductor and the real conducting layer is well below the surface. Then for either case the ground wires and the conducting layer of earth form a condenser of large capacity as compared to that between aerial and ground wires. The capacity between the aerial and the conducting layer of earth is then due to these two capacities in series and since one is

large as compared to the other it will be very nearly in value that of the smaller capacity, which is that of the antenna.

Choice of Wave Length.—A mathematical analysis of the expression for the attenuation of the wave shown in Fig. 73 will show that for given transmitting and receiving antennæ there is, for a distance X separating them, a best value of the wave length and hence of the frequency to be used in transmission.[1] This value is $\lambda = a^2X^2/4$. It thus appears that long waves (*i.e.*, low frequencies) are best for long distances. With allowance for this fact the available range of wave length is divided up and assigned to different groups of stations, *e.g.*, ship stations, navy stations, long-haul commercial land stations.

Wireless Telegraphy.—In the previous pages there have been discussed the methods of producing high-frequency alternating currents, the methods of detecting their existence and the phenomena of the transmission of these effects through the ether. If distances of only a few hundred miles are involved it is usual to use either a spark-gap transmitter or a vacuum-tube transmitter, to work with short wave lengths, *e.g.*, 200 to 1000 meters, and to receive with a crystal detector or a vacuum tube. For longer distances the tendency has been to use arc generators or alternators, to use longer wave lengths and to receive heterodyne with a vacuum tube. The choice of apparatus, its design and the circuit in which it is connected has been influenced in the past largely by question of patent rights and by the individual bias of the designing engineer. There are a large number of circuit arrangements all about equally good but frequently unnecessarily complicated. The safest rule is to adopt the simplest and most flexible circuit. Thus for transmission the loosely (inductively) coupled tuned circuits of Fig. 42 will be found satisfactory. For reception the circuits of Fig. 31 or Fig. 35 may be used. Some alterations of these fundamental circuits will, of course, prove desirable;[2] thus in Fig. 35 it may be best to supply the plate

[1] Using Austin's data, $a = 0.0015$ when X and λ are in kilometers.

[2] Illustrations of practical circuits will be found in the next chapter.

battery in shunt with the receiver instead of in series. If this is done a condenser should be put in series with the receiver to keep out of it the battery current and a choke coil or high reactance should be put in series with the battery so that it may not serve to short circuit the receiver for currents of audio-frequency.

Wireless Telephony.—In wireless telephony a continuously transmitted high frequency is modulated by the voice frequency. This is accomplished by causing the currents in a telephone transmitter circuit to affect the high frequency as it is being produced. Thus in Fig. 79 is shown an arrangement whereby the voltage impressed on a Poulsen arc is caused to fluctuate in value by superimposing upon it a voltage induced by the telephone circuit.[1]

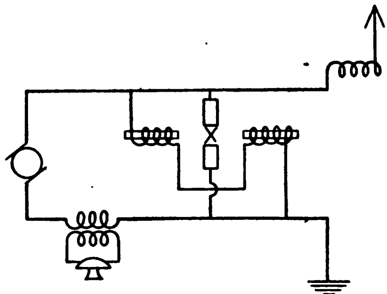


FIG. 79.—Poulsen arc as a radio telephone transmitter.

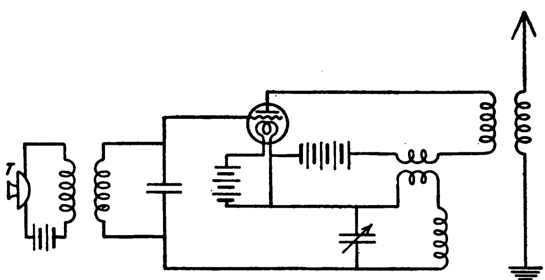


FIG. 80.—Oscillating vacuum tube as a radio telephone transmitter.

[1] The method most frequently adopted is the exact analogue of wire telephony. One or more transmitters are connected directly into the antenna circuit. The transmitter thus serves to modulate the high frequency (or carrier wave) in exactly the same way as the transmitter in wire telephony modulates the direct current or zero frequency carrier. In this connection see the following section.

Fig. 80 shows how the current in a telephone transmitter circuit may be impressed on a vacuum-tube generator.

The general principle involved is the same in the two circuits, but is more easily grasped in the case of the vacuum-tube generator because of its simpler equations. There is impressed upon the input of the tube of Fig. 80 two voltages, one a continuous, high frequency, say $A\epsilon^{j\omega_0 t}$, and the other a low or audio-frequency, say $B\epsilon^{j\omega_1 t}$. The output of the tube may then be found by trigonometry similar to that used in discussing heterodyne receiving in connection with Fig. 35. It is thus seen that there are transmitted two high frequencies, one $(\omega_0 + \omega_1)/2\pi$ and the other $(\omega_0 - \omega_1)/2\pi$. Also, of course, the tube acts as an amplifier or repeater and so transmits directly the high frequency $\omega_0/2\pi$. At the receiving station then the current in a detector would comprise components having frequencies of the sum and the difference of those it receives. It would thus produce in its output circuit a frequency of $\omega_1/2\pi$ which is the frequency of the talk input at the transmitting station.

It will thus be seen that, as was pointed out on page 56, so far as the principle on which heterodyne receiving systems operate is concerned, it is immaterial whether the two superimposed high frequencies are produced one at the sending and one at the receiving station or both at the sending station as in wireless telephony.

Transmission of Intelligence.—It is now possible to make a comprehensive statement of electrical methods of transmitting intelligence. Two frequencies may always be considered to be transmitted and received. One of these may be denoted the carrier frequency, and the other the modulating frequency. In wire telephony, a direct current or zero frequency current serves as the carrier. This in passing through the transmitter button is modulated by an audio-frequency. The introduction of a transformer or so-called "repeating coil" between the transmitter circuit and the line reduces the amplitude of the zero frequency current in the line to zero. In wire telegraphy, a

direct current is modulated by a complex wave of audio-frequencies. In wireless telephony, as we have just seen, a carrier frequency is modulated by an audio-frequency resulting in the transmission of three radio-frequencies. In wireless telegraphy, using continuous waves, the carrier frequency may be considered to be modulated by an imaginary current of frequency determined by the conditions at the receiving station. Thus, in heterodyne receiving with a separate generator, or with an oscillating audion, this frequency is determined by that local source. With a tone wheel or tikker, the frequency is determined by that instrument. In wireless telegraphy, using spark gaps a continuous frequency is modulated by the group frequency of the system. In all these cases there may be considered to be a fundamental principle of which wireless telephony is the general case and the other special cases.

Along the same line, it may be pointed out that wire communication is the special case of which wireless is the general case. If waves arising from a transmitting station are restricted and guided by continuing the horizontal parts of the two antennæ until they meet, the directive system thus formed becomes the familiar circuit of telegraphy, or the early days of telephony, consisting of a single wire and the ground return. Replacing the earth by a wire to form a self-contained system with immensely increased possibilities of reducing interference from outside sources was first suggested by Col. Carty. The use of a high frequency for the carrier instead of the zero frequency commonly used in wire telephony has been suggested by Major Gen. Squier.

The student then, who grasps the fundamental ideas involved in modulation and detection as illustrated by wireless telephony, starts from what may be considered to be theoretically, although it is not historically, the basis of all systems of communication by electromagnetic waves.

CHAPTER VIII

PRACTICAL APPLIANCES AND METHODS OF RADIO TELEGRAPHY

Resistances.—When a conductor is traversed by an alternating current of high frequency, it is found that the resistance of the conductor depends upon the frequency. The factor by which the direct-current resistance of the wire must be multiplied to give the alternating-current resistance is called the "skin effect

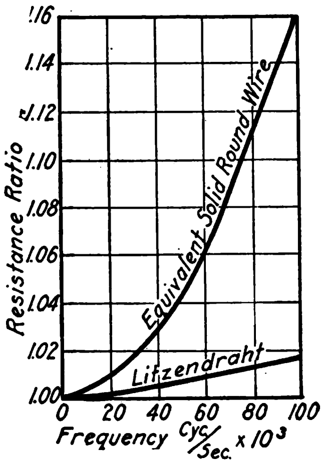


FIG. 81*a*.

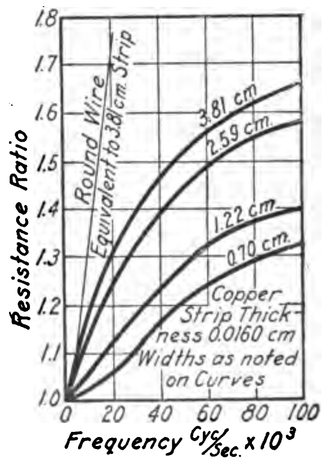


FIG. 81*b*.

FIGS. 81*a* AND 81*b*.—Skin effect.

resistance ratio." This name is due to the fact that the higher the frequency, the smaller the current density at the center of the wire and the larger the portion of the current carried by the outer layer or skin. The effect is due to the distortion of the

lines of current flow by the alternating magnetic field established by the current. The ratio defined above increases with the size of wire used.

Replacing a solid conductor by several strands of wire having the same total cross-section will reduce this ratio if the strands are insulated and braided so that each is on the outside surface for about the same portion of the total length. In the case of the so-called "Litzendraht" wire, the reduction in skin effect is very marked[1] as is evident from Fig. 81a. This wire is made up of a large number of small strands enameled for insulation and braided together with a so-called "basket weave" into a cylinder.

A very convenient method of securing a substantial reduction in skin effect over that for a solid cylindrical wire of the same area is to form the conductor in thin and narrow strips or ribbons. This is well evidenced in Fig. 81b.

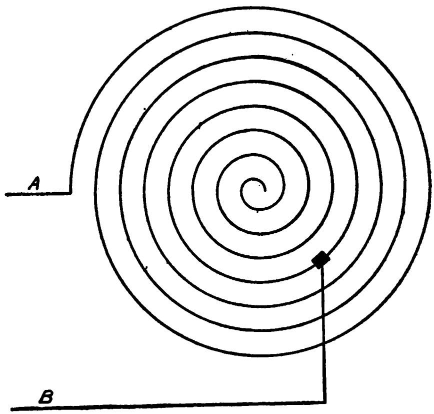


FIG. 82.—Spiral inductance.

In the construction of coils the best results seem to have been obtained by using either copper ribbon or Litzendraht. In the earlier days of the art, coils were sometimes formed from hollow tubing, but the ribbon construction is superior.

Inductances.—Inductance coils are usually wound as solenoids of one or more layers or as flat spirals. The latter construction is much used in small quenched-gap transmitting sets. In such cases, the coils are frequently of the form shown in Fig. 82. One connection is made at *A* and the other by a movable clamp at any desired point; or both connections may be so adjustable.

[1] KENNELLY AND AFFEL: *Proc.* I. R. E., vol. 4, pp. 523–75, 1916, from which article Figs. 81a and b are reproduced.

Multi-layer coils result in an increased inductance for the same space occupied by the coil but are subject to especial difficulties of design. Between any two turns of an inductance coil carrying an alternating current there is a potential difference which is greater the larger the number of intervening turns. In winding a multi-layer coil, turns which are separated electrically by several intervening turns may be brought into close proximity. This results in increased difficulties of insulation and in increased energy losses in the insulation. Furthermore, between any two turns of a coil there is capacity. At low frequencies this so-called "shunted capacity" of a coil may be inappreciable but at high frequencies it may become quite objectionable. Since the capacity between two conductors is inversely as the distance between them, these effects may be much increased in multi-layer construction. In the present state of the art, it is doubtful that the decrease in size of coils effected by multi-layer winding outweighs for ordinary purposes the inherent disadvantages.

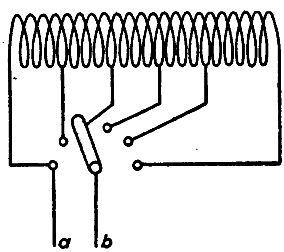


FIG. 83.—Inductance variable by steps.

Variable Inductances.—Variations in inductance may be produced by fixed steps or continuously. As an illustration of the first case see Fig. 83 where a dial switch is so arranged as to vary the inductance included between *a* and *b*. Variations of smaller steps may be obtained by using a continuous single-layer coil with a sliding contact as shown in Fig. 84.

Continuous variations are usually accomplished by using two coils, one fixed and one moving, so that their mutual inductance may be varied. In this way the inductance of the combination in series is changed from a maximum when the coils are aiding to a minimum when their fields are opposing. The relative motion of the two coils may be accomplished in any of several ways. Thus in Fig. 85, coil II may slide into the larger coil I. A mini-

mum inductance with this design is to be obtained by turning coil *II* end for end and then inserting it in *I*.

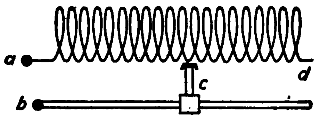


FIG. 84.—Slide-wire inductance.

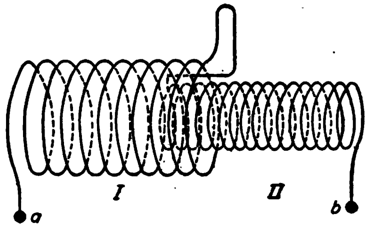


FIG. 85.—Series inductances with variable coupling.

The most satisfactory method is to use some form of "variometer" as that shown in Fig. 86 where one coil is placed inside

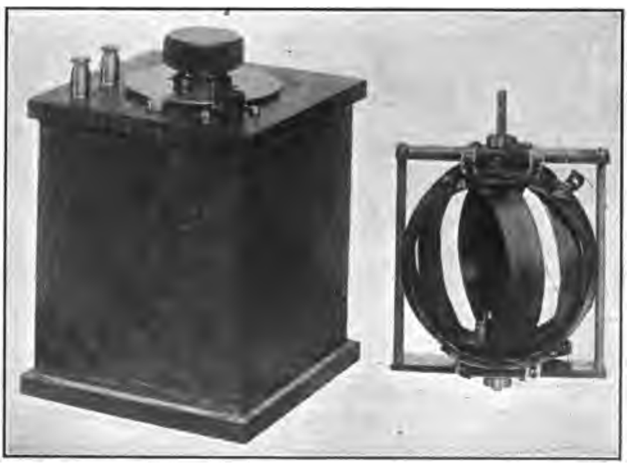


FIG. 86.—Variometer for small currents.

the other and the plane of the inside coil may be rotated about a diameter.

With the spiral coil of Fig. 82 continuous variations may of course be obtained by sliding one of the clips or contacts. In all such cases where a variometer is not used, it is important to

guard against "dead ends." Thus in Fig. 84 it will be seen that the current flowing in the section *acb* induces an e.m.f. in the dead end *cd* as in an auto-transformer. If the shunted capacity of the whole coil is such as to give it a natural frequency near that of the current large energy losses will ensue.

Condensers.—Condensers as used in wireless operation divide sharply into two classes, namely, those used in transmitting systems and those used in receiving systems. In the latter case only small potentials are met and the usual practice is to con-

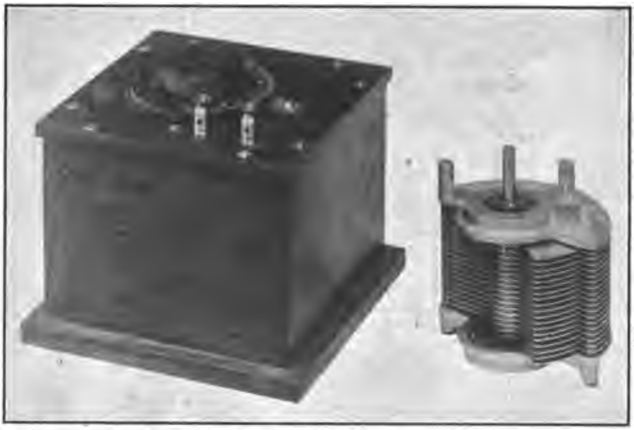


Fig. 87.—Variable air condenser.

struct fixed condensers with mica for a dielectric and variable condensers with air for the dielectric. The latter case is illustrated in Fig. 87. The condenser consists of two sets of semicircular plates, having a common axis and alternately spaced. One set is fixed and the other movable. The maximum capacity occurs, of course, when the plates of one set are immediately above those of the other.

In transmitting systems large e.m.f.'s are involved and hence leakage discharges and punctures due to the breaking down of the dielectric must be guarded against. In many instances glass-

plate condensers are used, the conducting plates being thin sheet metal or foil. In other cases condensers formed by sets of alternate conducting plates have been enclosed in containers filled with air or other gases under pressure, or with oil. Fig. 88 shows an external view of a compressed-air condenser. Fig. 89

FIG. 88.—Compressed-air condenser. FIG. 89.—Inside of compressed-air condenser.

shows the inside construction. The so-called Leyden jar is also used but its volume is large for the capacity it has as compared to plate condensers. The Leyden jar consists merely of a glass jar which to a height somewhat below the top is coated inside and out with a metal film. The two metal surfaces constitute the conducting plates of the condenser.

In connecting condensers it should be remembered that for condensers in parallel the resultant capacity is the sum of the separate capacities but that for condensers in series the resultant capacity is found by taking the reciprocal of the sum of the reciprocals of the several capacities.

Frequency Measurements.—The phenomenon of resonance as discussed in Chapter V is the basis of the convenient methods for determining frequency. Given a calibrated variable condenser and a source of current of a known frequency the inductance of a coil may be determined as shown on page 86. The coil and condenser may then be used to form a frequency meter as described on page 84. The minimum requirement of apparatus for measurements of high frequency is then a fixed and known inductance, a calibrated variable condenser and a detector. In amateur installations where equipment is lacking, it is usual to calibrate a few points of the wave meter directly in terms of frequency or wave length by tuning it as a receiving circuit to various transmitting stations of known wave lengths.

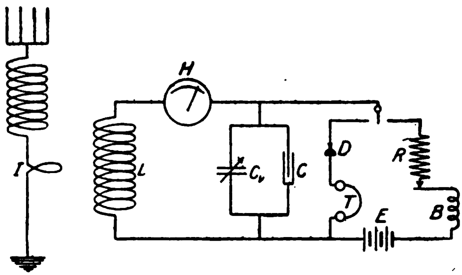


FIG. 90.—Wave-meter circuit.

It is also possible to wind certain forms of coils very exactly to predetermined values of the inductance. The calculations[1] are, however, usually laborious and beyond the amateur for whose purposes approximate results may be obtained by winding cylindrical coils which are long as compared to their diameters.

[1] ROSA: *Bull.* Bureau Standards, vol. 8, pp. 1–237, 1911.

For such a coil the inductance[1] is $4\pi^2r^2N^2/l$ where l, r, and N are the radius, the length and the total number of turns respectively.

Wave Meters.—Commercial forms of frequency or wave

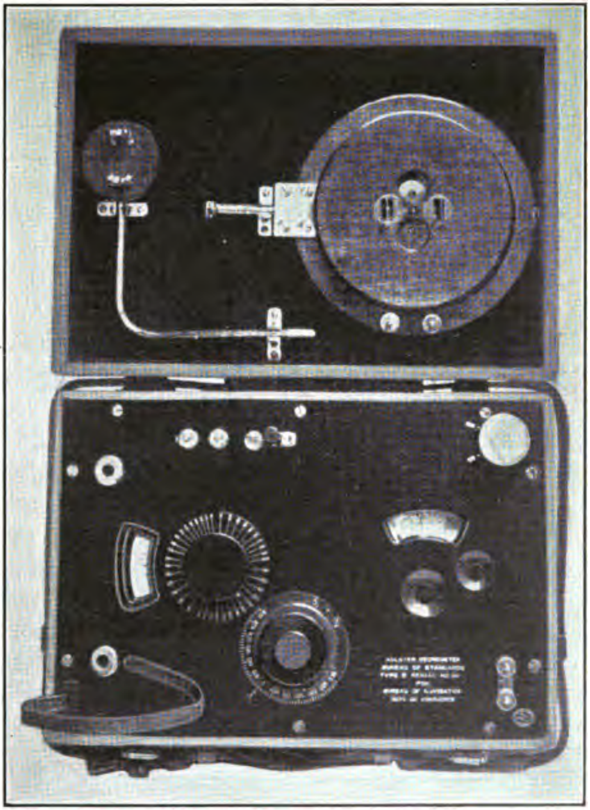


FIG. 91.—Kolster wave meter.

meters operate as illustrated in Fig. 90[2] upon the same principle as has been mentioned above. They usually embody also

[1] In centimeters if r and l are in centimeters. To reduce to henries divide by 10^9.

[2] Kolster: "Direct Reading Decremeter and Wave Meter." *Proc.* I.R.E., vol. 3, pp. 29–53, 1915. From this article Fig. 91 is reproduced.

the buzzer circuit described on page 85. In the figure, H is a hotwire ammeter and D is a crystal detector. The tuned circuit is formed by L and the condensers C and C_v the latter being variable. A switch permits the use of the buzzer circuit EBR where R is a resistance. The special features of such a set in addition to the convenient compactness as shown by the picture of Fig. 91 lie in the fact that its scales are calibrated to read directly not only wave lengths but also decrements. This last is obtained by the use of a specially designed variable condenser the plates of which are as shown in Fig. 92. Such a condenser has a capacity of $a\epsilon^{mD}$ where a and m are constants and D is the angle through which the plates are rotated. The advantage of this design lies in the fact that in moving the plates through any angle the ratio of the change in capacity, $C_r - C$, to the new value C of the capacity is a constant. This makes the decrement of the circuit depend upon the angle through which the condenser plates are rotated and hence by a system of gears permits a direct reading of the decrement of the circuit under measurement.

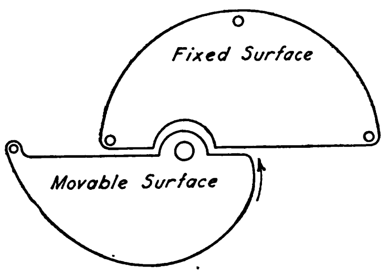


FIG. 92.—Condenser plates of Kolster meter.

Measurement of Logarithmic Decrement of Wave Meter.— The decrement of a wave-meter circuit is known if its resistance and either the inductance or the capacity are known. This follows from the relations of equation (39), namely $d = \frac{a}{f}$ where for a circuit of the form under consideration, a has been shown to be $\frac{R}{2L}$.

Thus

$$d = \frac{R}{2Lf} = \frac{R\pi}{L\omega} = \frac{R\pi}{\sqrt{\frac{L}{C}}} = R\pi\omega C \qquad (67)$$

If the wave meter is excited by a buzzer and causes a deflection corresponding to an effective current of I in a hot-wire ammeter included in its circuit, then the heating effect of the natural oscillation is I^2R where R is the resistance of the wave-meter circuit at this frequency. Now let the resistance be increased by adding a known resistance R_1, then under the same conditions for the buzzer it is correct, to a first approximation, to assume that the natural oscillations dissipate the same amount of energy per second. Hence $I^2R = I_1^2(R + R_1)$ where I_1 is the new value of the current indicated by the hot-wire instrument.
Hence

$$R = R_1\frac{I_1^2}{I^2 - I_1^2} \tag{68}$$

It is usual to insert resistance by a slide wire sufficient to make $I_1^2 = \frac{I^2}{2}$. Equation (68) then becomes $R = R_1$.

The method just described is convenient in that it requires no apparatus which is not normally part of a wave-meter set. A more exact method will be described at the end of the next section.

Bjerknes Method of Decrement Determination.—It has been shown by Bjerknes that if a resonance curve is taken for an input of unknown decrement d_1 by varying the capacity of the wave meter, then, if the decrement d_2 of the wave meter is known, d_1 may be determined from the curve. Thus consider the resonance curve of Fig. 93. The ordinates are proportional to the heating effect of the current as indicated by the hot-wire ammeter of Fig. 90. The abscissas may be the ratios of the capacity C_x in circuit to the capacity C_r for which resonance occurs or they may be actual values of the capacity as indicated in the figure.

Then the relation

$$d_1 + d_2 = \pi\frac{C_r - C_x'}{C_x}\sqrt{\frac{I_x^2}{I_r^2 - I_x^2}} \tag{69}$$

holds, provided d_1 and d_2 are small as compared to 2π and $\frac{C_r - C_x}{C_x}$

is small as compared to unity. If C_x is greater than C_r, then this factor is written $\frac{C_x - C_r}{C_x}$ so that it is always positive since the decrement is positive. In exact work it is usual to average determinations from both sides of the curve.

The usual practical method is to determine but three points on the resonance curve, namely those corresponding to I_r^2, $I_1^2 = \frac{I_r^2}{2}$ and $I_2^2 = \frac{I_r^2}{2}$. In terms of the corresponding capacities the formula then becomes

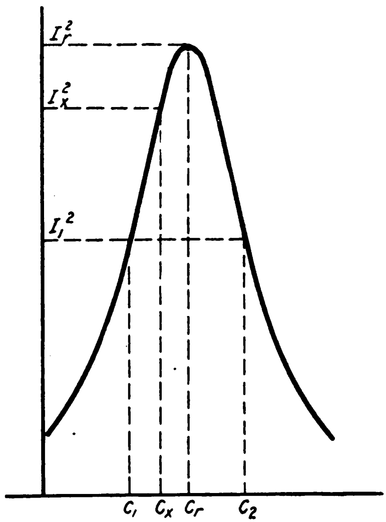


FIG. 93.—Determination of decrement from resonance curve.

$$d_1 + d_2 = \pi \frac{C_2 - C_1}{C_2 + C_1} \qquad (70)$$

The Kolster decrement meter makes use of this relation. The condenser is varied until the value of I_r^2 is noted It is then decreased to the value C_1. The gears of an auxiliary scale giving $d_1 + d_2$ are now meshed with a gear wheel on the condense. axle. A pointer is then set opposite the zero of the decrement scale. The condenser is rotated until C_2 is reached in which position the pointer indicates d_1.

It is now evident why it is necessary to know the decrement of the wave-meter circuit in order to determine the decrement of the impressed e.m.f. It is also evident how the decrement of the wave meter may be determined provided a source of sustained oscillation is available of the same frequency as the e.m.f. of unknown decrement. Thus, tune the wave meter to the unknown e.m.f. Then couple the wave meter to a generator of an undamped e.m.f. and tune the generator to the same frequency

as indicated by a maximum effect in the wave-meter circuit. Determine $d'_1 + d_2$ for the input from the generator by the Bjerknes formula. Since, however, d'_1 is zero this determination gives at once the decrement of the wave-meter circuit at the desired frequency.

Decrements and Frequencies for Circuits of Two Degrees of Freedom.—On page 84 it was shown that two natural oscillations occur in a circuit of two degrees of freedom, such as that of a spark transmitter coupled to an antenna. If the primary and secondary are "syntonized," that is if $L_1C_1 = L_2C_2$ as in the equations of page 76 then the current in each circuit consists of two components, one of frequency above and the other

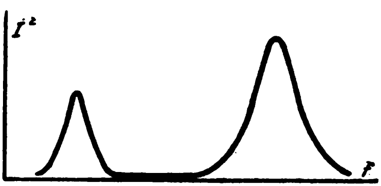


FIG. 94.—Resonance curve for closely coupled circuit.

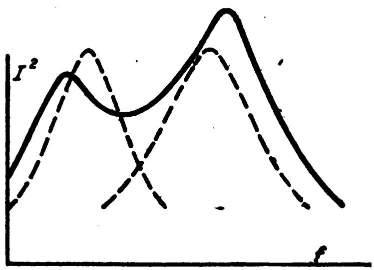


FIG. 95.—Components in double-peaked resonance curve.

of frequency below that for which the circuits are separately tuned. To each frequency there corresponds of course a damping constant. In case the coupling is close these two frequencies are widely separated. A resonance curve taken by coupling a wave meter loosely to either of the oscillating circuits will then be of the form shown in Fig. 94. For the case where the peaks are far apart each peak may be treated as a separate curve and the frequency and decrement of the corresponding oscillation may be determined by the methods previously described.

If the peaks are fairly close together as in the full line curve of Fig. 95 such treatment is not justified, since both components

contribute to the effect in the wave meter. Some indications may, however, be obtained by making use of an approximate relation which exists between the currents in the primary and secondary circuits. In the case of one of the component oscillations the current in the secondary is practically in phase with that in the primary, but for the other component the two currents are practically 180° out of phase. If then a current loop is introduced into each circuit as in Fig. 96 and if the wave meter is loosely coupled to these loops as shown, then either component oscillation may be balanced out so far as its effect on

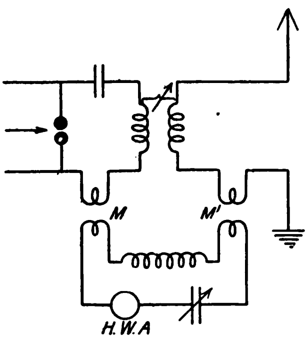


FIG. 96.—Wave meter circuit for analyzing Fig. 95.

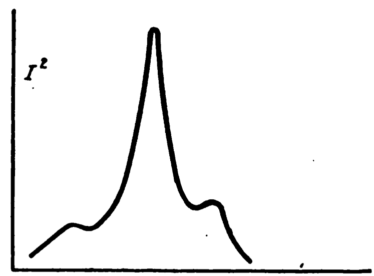


FIG. 97.—Resonance curve for transmitter with incomplete quenching.

the wave meter. The adjustment is of course made by varying the coupling between the wave meter coils and either the primary or the secondary. The component thus balanced out may be introduced and the other component eliminated by reversing one of the coupling coils of the wave meter. The resonance curves for the separate components will then be of the form shown by the dotted curves of Fig. 95.

In the case of quenched gap operation the resonance curve may show three peaks. In that event there should be one prominent peak corresponding to the frequency of the natural oscillations of the secondary after quenching has occurred in

the primary. Before quenching occurs the conditions are as discussed above. The more complete the quenching the smaller will be the energy corresponding to the oscillations of the circuit as one of two degrees of freedom, and hence the smaller the corresponding peaks in the resonance curve. A curve illustrating these conditions is shown in Fig. 97.

Antenna Constants.—The constants of a given antenna may be found in several ways. The methods developed earlier involved the use in the antenna circuit of a spark gap by which oscillations were produced. These oscillations were then received in a wave-meter circuit.

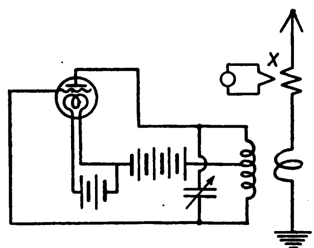


FIG. 98.—Circuit for measuring constants of antenna.

The most convenient methods today are those using some generator of sustained oscillations, as for example, a vacuum-tube generator. A current-measuring instrument may be connected in the antenna circuit and the latter treated as a wave meter. Its resonant frequency and decrement may then be determined by the usual methods. The scheme of connections using a thermo-couple X as in Fig. 98 may be thus employed.

If L_0 and C_0 are the effective inductance and capacity of the antenna, then its natural frequency, f_0, is given by

$$f_0 = \frac{1}{2\pi\sqrt{L_0C_0}} \tag{71}$$

The values of L_0 and C_0 may then be found by introducing into the antenna a known amount of inductance L_1 or of capacity C_1. The resonant frequency will thus be changed. If the new value is f_1 then we have

$$f_1 = \frac{1}{2\pi\sqrt{(L_0 + L_1)C_0}} \tag{72}$$

or

$$f'_1 = \frac{1}{2\pi\sqrt{L_0 C_x}} \tag{73}$$

Where

$$\frac{1}{C_x} = \frac{1}{C_0} + \frac{1}{C_1}$$

By eliminating L_0C_0 from equations (71) and (72) the capacity C_0 is obtained as

$$C_0 = \frac{1}{4\pi^2 L_1}\left(\frac{1}{f_1^2} - \frac{1}{f_0^2}\right) \tag{74}$$

and by substitution in equation (71) L_0 is of course obtained. Similar relations may be established with equation (73).

In case, as sometimes happens, the inductance introduced into the antenna circuit by the coupling coil shown in Fig. 98 is not negligible, then the conditions are represented by an equation similar to (72) above. By adding more inductance, which must not be coupled to the generator, a new relation is obtained. From these two equations C_0 and L_0 may be obtained and hence also f_0 by using equation (71).

Antenna Loading.—It is usual to increase the wave length to which an antenna is tuned by loading. In this case, as indicated by equation (72), series inductance is inserted. On the other hand, if it is desired to tune to a wave length shorter than the natural wave length of the antenna it is necessary to load with series capacity as is indicated in equation (73).

In the construction of transmitting or receiving sets it is necessary, then, to know in advance the range of wave lengths over which the set is to operate, the natural frequency of the antenna and its effective capacity. With this data it is possible to determine by equations (72) or (73) above, the type and amount of loading for the antenna circuit which the set must contain.

When it is desired to utilize an antenna for receiving at any wave length over a fairly large range without readjusting the loading, it is usual to load the antenna with resistance. In this

case the resistance is not directly inserted but is effectively inserted, as may be seen from equation (58) of page 79, by coupling the receiving circuit very closely to the antenna. The resonance curve of the antenna is then much flattened out. Although the efficiency is reduced it is rendered nearly uniform over a large range of frequencies. A receiving circuit so connected is said to be in a "listening" or "standby" adjustment.

Circuits Involving Vacuum Tubes.—The various "characteristics" of a vacuum tube have been discussed in Chapter III. From such static characteristics data is to be obtained as to conditions of current and voltages under which a given tube is

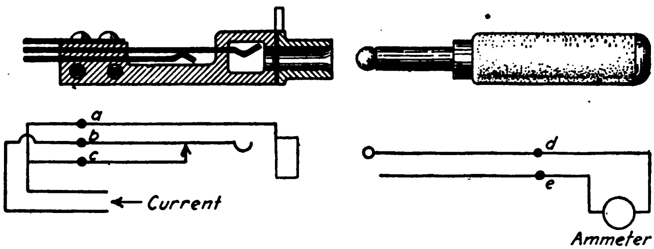


FIG. 99.—Plug and jack for connection of measuring instruments.

to be operated in order to obtain a desired effect. Except therefore under conditions where economy of space and weight are of prime importance any installation involving vacuum tubes should provide ammeters and voltmeters of proper range for determining the values of the various direct currents and e.m.f.'s. The values for best operating conditions and the limits which should not be exceeded are usually specified by the manufacturer of the tube. Where measuring instruments are available it is always possible to obtain the characteristic curve of a tube in the set in which it is to operate. In case a set is not operating normally this method may be of great value in locating the fault.

Where current measuring instruments are provided it is good practice to connect them into circuit through switches by which

the instruments may be short-circuited when not in actual use. A convenient form of switch for this purpose is furnished by the jack and plug of telegraph and telephone engineering. Thus Fig. 99 shows a jack with two points short-circuited and the plug by which an instrument may be inserted. If it is desired to make connections to a voltmeter then the circuit is connected to terminals *a* and *b* of the jack and terminal *c* is not used.

In Chapters III, IV and VII it has been shown that the vacuum tube possesses a characteristic adaptable to a large variety of purposes. Thus it may act as an amplifier, that is, as a repeater. If its input and output are coupled it acts as a

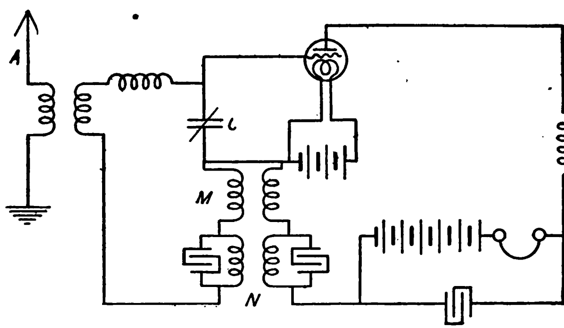


FIG. 100.—Oscillating vacuum tube detector with feed back for amplification.

generator. If its amplification is accompanied by distortion it acts as a detector or as a modulator. It may also be used as a rectifier or, as may be seen from a study of its characteristic, as an automatic current limiting device.[1] Some of these operations may occur simultaneously, as for example, detection and amplification, or generation and detection.[2]

In case a single tube performs two or more operations it is of course possible to arrange a circuit equivalent so far as these

[1] As for example in the circuit arrangement of Arnold, U. S. Patents Nos. 1,168,270 and 1,200,796.

[2] As for example in some of the receiving circuits devised by DeForest; or c. f. the circuit of Fig. 100.

operations are concerned in which two or more tubes are involved, each, however, performing but one type of operation. Thus consider the circuit[1] shown in Fig. 100 where there is a generation of high frequency oscillations, an amplification of the input from the antenna, a detection of the superimposed signal and heterodyne currents, and an amplification of the resulting audio-frequency beat note. For purposes of analysis and of quantitative laboratory experiment this circuit may be replaced by the circuit represented schematically in Fig. 101 where A_1 and A_2 represent vacuum tube amplifiers of the impressed high frequency and of the audio-frequency beat note,

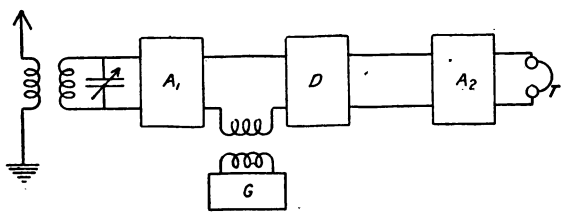


FIG. 101.—Equivalent of Fig. 100.

respectively, G represents a vacuum tube generator and D represents a vacuum tube detector. The rectangle in each case represents the vacuum tube and its associated apparatus.

The circuit of Fig. 101 involves vastly more equipment than that of Fig. 100. It admits, however, of the use in each operation of the tube and of the circuit best adapted for each purpose.

Grid Circuit Condenser.—In vacuum tube circuits there is sometimes inserted a series condenser between the grid and the receiving circuit. Such a condenser is not necessary for operation as a detector if the various e.m.f.'s in the circuit are such as to bring the tube to the proper point on its characteristic curve. If, however, the grid is uncharged, that is if E_C is zero,[2] then the

[1] Described by Armstrong c. f. "Some Recent Developments in the Audion Receiver." I.R.E. vol. 3, pp. 215–38, 1915.

[2] For function of E_C see Figs. 25 and 27 and text of pp. 45–48.

tube may or may not operate efficiently, depending upon the form of its characteristic as determined by its constants and by the values of I_A and E_B. The introduction of the grid circuit condenser causes detector action by its blocking or trapping effect. Thus imagine that the first half wave of the input radio e.m.f. makes the grid negative with respect to the filament. This half wave of e.m.f. tends then to force electrons from grid to filament, but since electrons are not emitted by the grid, no transfer takes place. Making the grid negative reduces, however, the current in the plate circuit. The succeeding half wave of the input tends to force electrons in the opposite direc-

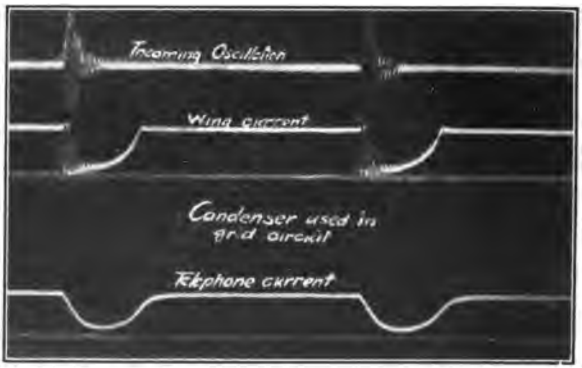


FIG. 102.—Oscillograms of grid e.m.f. and plate current.

tion, namely, from filament to grid and since this is the possible direction of transfer, such action occurs. The electrons so transferred to the grid are, however, blocked in their passage through the wire circuit connecting grid and filament by the condenser. The result of this unilateral conductivity of the grid-filament circuit combined with the trapping action of the condenser is that the grid becomes more negative with each succeeding wave of the input e.m.f. and the current in the plate circuit decreases. If the e.m.f. consists of a train of waves as shown in Fig. 102 then the plate (*i.e.*, so-called wing) current is as

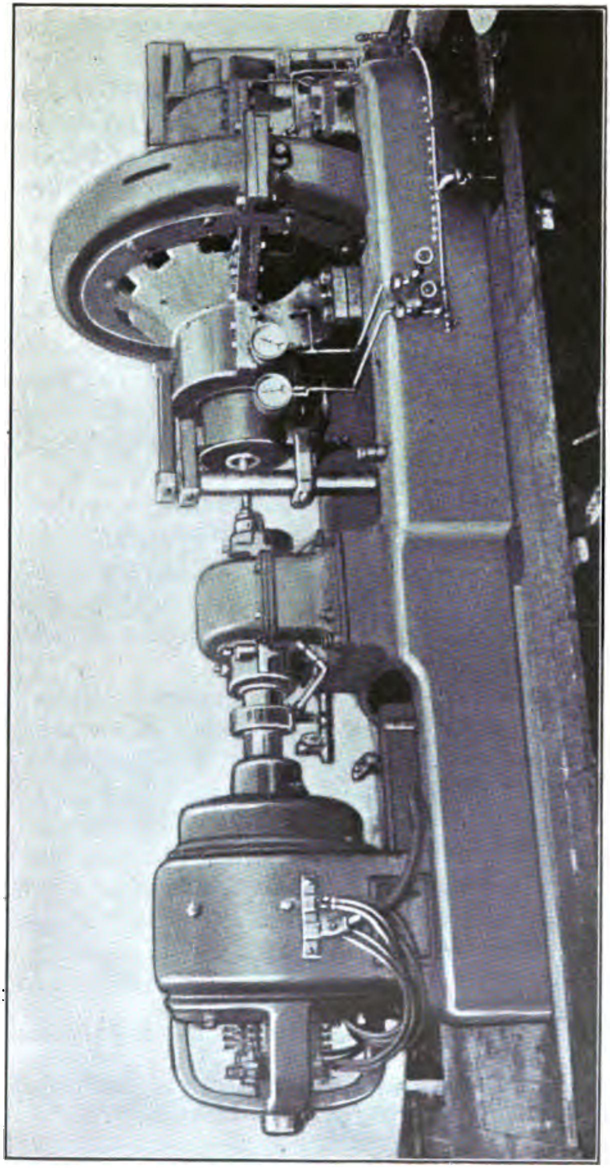


FIG. 103.—Alexanderson alternator.

shown in the oscillogram.[1] The audio-frequency current in a receiver connected in the plate circuit, is as shown in the bottom curve of the figure. It is evident that between one train of waves and the next the trapped negative charge on the grid must be allowed to leak off. For tubes with some gas or with a leaky condenser the grid gradually returns to its normal condition. To provide such a leak in the case of vacuum tubes it is usual to connect a resistance of several hundred thousand ohms either across the condenser or directly from grid to filament.[2]

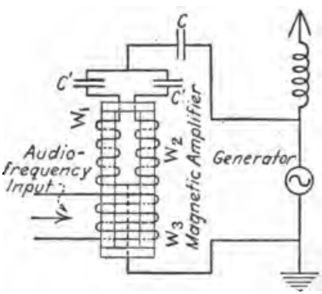


FIG. 104.—Simplified circuit of magnetic amplifier.

It is now evident that if no conducting path is offered from a grid to its filament, that is if the grid is "floating," it may "pick up" a negative charge. It is also evident that an excessive negative charge occasioned, for example, by atmospheric disturbances may make the grid so negative as to render the tube inoperative for an appreciable time.

Magnetic Amplifier.—One of the obvious methods for the modulation of a current by a second source of current is illustrated by the dependence in a generator of its terminal voltage, and hence, other things being equal, of its output current upon its field excitation. If the variations in the field are to be of the frequency of the human voice, as would be necessary in an amplifier of telephone currents or in a modulator for wireless telephony, the losses due to hysteresis and eddy currents in the iron core become prohibitively large. As the reader, who is familiar with power engineering, already knows the output

[1] Reproduced from the paper of Armstrong, (l. c.) who first published the explanation of the phenomena under discussion.

[2] The time required for a condenser to leak off a given portion of its charge may be found by applying the methods of Chap. V.

voltage of an alternator is extremely sensitive to the character of the load connected to it and to the amount of current supplied. It is further evident that if the load is inductive variations in the terminal e.m.f. of the alternator may be occasioned by varying the inductance. This variation may be accomplished by varying the permeability of the iron.[1]

The development from these more or less well-known principles of a complete and efficient system for the modulation of large power outputs of high frequency alternators by audio-frequencies was accomplished by Alexanderson.[2] The circuit which he devised is shown in Fig. 104. A high frequency alternator is connected directly into the tuned antenna circuit. Shunting this alternator is his magnetic amplifier. The alternator used in the tests which the inventor reports is shown with its motor drive and gearing in Fig. 103. The magnetic amplifier is shown in Fig. 105.

Returning to Fig. 104 it is seen that the alternator is shunted by a circuit containing a condenser C in series with an iron cored inductance which has two parallel and opposite windings. These windings are made open circuits so far as audio-frequencies are concerned by the introduction of two condensers, C', of small impedance to radio-frequencies. This prevents the circulation in the windings W_1 and W_2 of any audio-frequency current induced by winding W_3 the effect of which would be to oppose the changes in flux which W_3 is to accomplish. To the winding W_3 is supplied the audio-frequency with which it is desired to produce modulations in the e.m.f. supplied by the alternator to the antenna. The series condenser C is chosen to neutralize the inductance of W_1 and W_2 (for some definite value of the direct current in W_3) and results in a greater sensitiveness in

[1] The method of producing a double frequency current from iron core transformers as described on page 100, is an example of the principle involved in varying the permeability.

[2] cf. ALEXANDERSON: "A Magnetic Amplifier for Radio Telephony" *Proc.* I. R. E., vol., 4, pp. 101–120, 1916.

the control of the alternator output e.m.f. In the final form described by the inventor there is added in shunt with the circuits W_1C' and W_2C' another condenser (not shown in the figure under discussion). With this condenser the normal or average e.m.f. impressed on the antenna is increased and the alternator is used to better advantage.

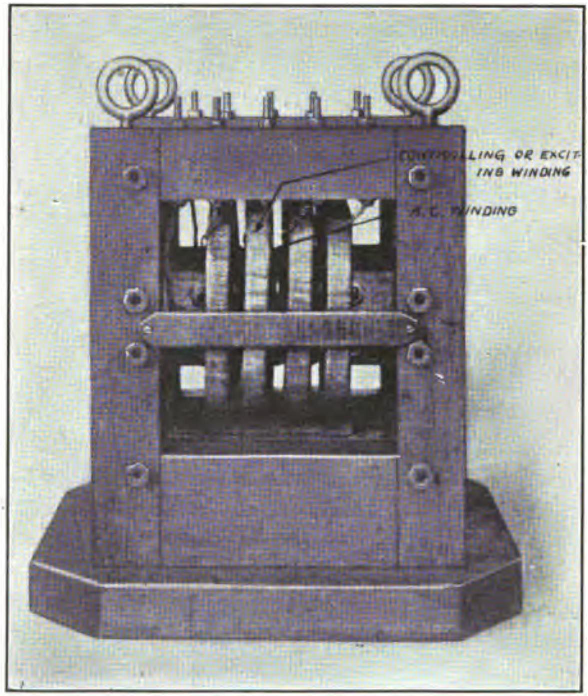


Fig. 105.—Alexanderson magnetic amplifier.

Audibility.—In the early days of the wireless art the need of a unit for measuring the intensity of received signals became apparent. The term "audibility factor" was introduced and is in common use today. This factor is defined as "the ratio of the telephone current producing the received signals to that

producing the least audible signal at the given audio-frequency." Thus in Fig. 106 the audibility of the signals received in the telephone is found by shunting across the receiver a non-inductive resistance S and reducing this resistance until the intensity of the signals just permits a differentiation on the part of the listener between dots and dashes. Then

$$A = \frac{\text{actual telephone current}}{\text{least audible current}} = \frac{Z_t + S}{S}$$

upon the assumption that the introduction of the shunt does not alter the audio-frequency current delivered by the detector. This condition is not always met in actual practice, particularly when using crystal detectors.

A further and more unwarranted assumption of common practice is to write $Z_t = R_t$ neglecting the reactance and the motional impedance of the telephone receiver. The audibility is then taken as $A = \frac{R_t + S}{S}$.

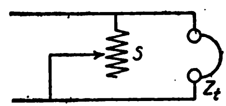


FIG. 106.—Circuit for measuring audibility.

To the student who is familiar with the characteristics of the telephone receiver as described in Chapter II, it is evident that widely different results may be obtained by the same observer if two sets of observations are made with two different receivers. In addition, the estimate depends upon the observer and upon physiological and psychological factors which are variable and indeterminate even for the same observer. The value of audibility determinations is therefore very small.

Accurate comparisons may, however, be made by the method of page 52. By using continuous wave generators such as the oscillating vacuum tube in connection with sensitive current measuring instruments, standards may now be constructed which were impossible in the earlier days of the art. No common agreement, however, as to a standard has, as yet, been reached.

Typical Transmitting Sets.—Figs. 107, 108 and 109 show in diagram the wiring connections for typical transmitting sets of

three types, namely synchronous rotary spark gap, quenched gap and Poulsen arc.

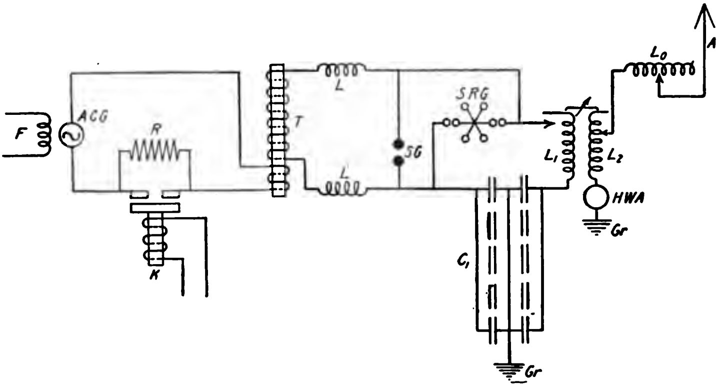


FIG. 107.—Typical synchronous rotary-gap transmitter.

In the case of the spark sets of Figs. 107 and 108 the alternator *ACG* supplies current to a transformer *T* of large magnetic leakage.[1] The transmitting key operates a relay *K* making or

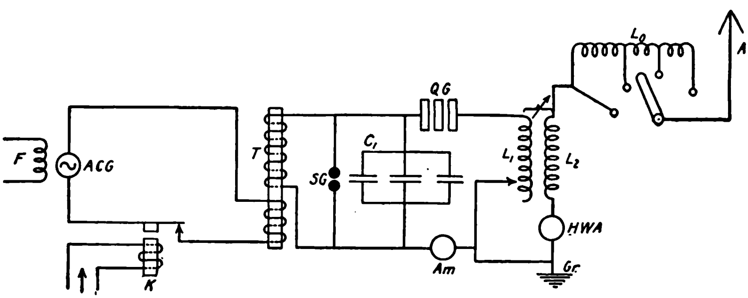


FIG. 108.—Typical quenched-gap transmitter.

breaking the circuit through the primary of the transformer.

[1] Consider the *T* equivalent of a transformer as developed on page 87. If there is large magnetic leakage the series inductances L_a and L_b are large and hence offer choking inductance to the currents in the primary and secondary.

In Fig. 107 the relay points are shunted by a resistance R to avoid excessive sparking. In this circuit additional choking inductance, L, has been added in the secondary. This still further reduces the coupling between the alternator and the spark gap circuit. In each circuit a safety spark gap SG appears which is set to break down at a voltage slightly higher than the normal operating voltage of the transmitting gap. A hot wire ammeter A_m may be included in the closed circuit, but the test of the radiation is the reading of HWA in the antenna circuit. The antenna loading coil L_0 may be set for fixed values

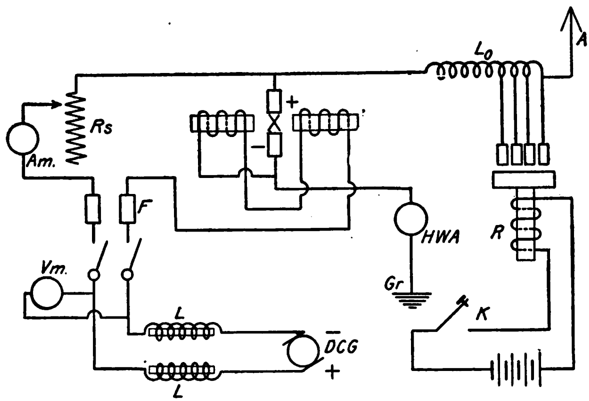


Fig. 109.—Typical Poulsen arc circuit.

as in Fig. 108 or may be more nearly continuous in its adjustment as in Fig. 107. Both primary and antenna circuits are grounded as shown at G_r.

In the arc set of Fig. 109 the generator DCG supplies current to the arc through protective choke coils, LL and a regulating resistance R_S. Fuses or a circuit breaker are installed at F. The arc is direct connected to the antenna through the loading coil L_0. In starting the arc, the electrodes are brought into contact and the arc struck at reduced voltage. The voltage and the arc length are then increased to the optimum value as

indicated by the hot wire ammeter *HWA*. Because the arc once struck must be maintained during transmission, the control can not be accomplished by a key or relay in the supply current circuit as in the preceding cases. The method, then, is to alter the wave length which is transmitted by short-circuiting some of the turns of the antenna loading. There is thus transmitted one wave length for dashes and dots and a different wave length during the time intervening between these signals. The difference may be only a matter of 100 meters in say 6000 corresponding to a frequency difference of about a thousand cycles. In tuning for a Poulsen arc station one may therefore pass through a position of tuning where the so-called "back stroke" or "compensation wave" is received.

Receiving Sets.—In the design of a receiving set provision must be made for loading the antenna so as to cover a large range of wave lengths without sacrificing sensitiveness. Beyond these common requirements commercial sets differ widely, first in the choice of a detector and second in the detector circuit. For ship to ship, or shore, communication with spark sets over short ranges a crystal detector may be used. For continuous wave detection a tikker or a tone wheel or the vacuum tube may be used. In the latter case the tube may be used with a separate source of current for heterodyne receiving or as an oscillating tube. If the oscillating tube is used the "feed back" may be either through conductive, inductive, or capacity coupling.

The variations in the inductance loading of the antenna may be occasioned solely by steps of various sizes or in part by a variometer. The coupling, if inductive, between the antenna and the detector circuit may be varied by rotating or sliding coils or by varying the portions of the inductance in either circuit, which are in a mutually inductive relation, and at the same time compensating by increasing the loading inductance in each circuit.

Relays may be provided for automatically protecting portions

of the receiving set when the antenna switch is changed from the receiving to the sending position. Resistances for varying the current to the detector, and ammeters for measuring such current may be included. A test buzzer circuit coupled to part of the antenna inductance is sometimes provided. Provision is also usually made for a "stand by" adjustment.

The type of service for which the set is designed, *e.g.*, ship or shore stations, commercial or military service, and the various types of the latter, determine or rather limit the design in weight

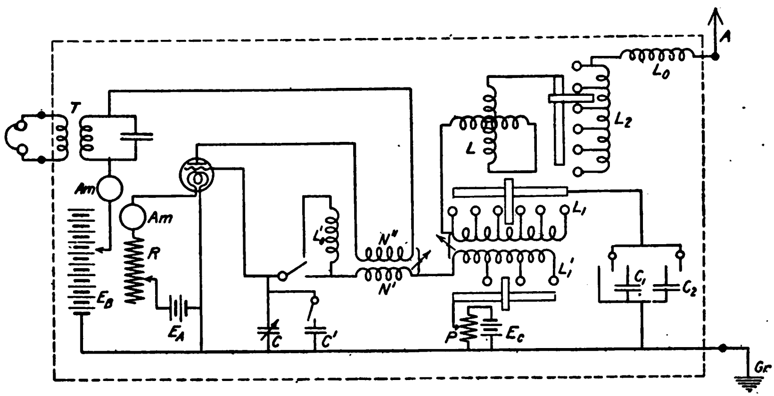


FIG. 110.—Vacuum-tube receiver.

and size, sharpness of tuning and sensitiveness. No attempt will therefore be made to cover the various sets that are in use today, but instead an illustration will be given of a possible set from which the fundamental principle may be seen. Thus consider Fig. 110.

The antenna loading for short wave lengths consists of two condensers, C_1 and C_2, either one or both of which may be in circuit. The antenna loading for the middle range of wave length consists of two inductance units, L_1 and L_2, which are variable by steps, one step of L_1 being a bit less than the total of the steps into which L_2 is divided. Finer variations may be

accomplished by a variometer, L. For the longer wave lengths a fixed inductance L_0 about equal to the maximum value of L_1 is added. Switches of convenient form for cutting L_0 into and out of circuit and also for introducing C_1 and C_2 are provided but not shown in the sketch. The inductance L_1 is inductively coupled to an inductance L'_1 in the detector circuit. The inductance in the latter circuit consists of the inductance L'_1 which is variable by steps and a loading inductance L'_0. The coupling between coils L_1 and L'_1 is adjustable. The tuning condenser C is variable and the total capacity of the detector circuit may be made variable to almost twice the maximum value of C by adding C' in parallel. A coil N' inserted in the detector circuit is coupled to a similar coil N'' in the output of the detector. An e.m.f. of E_C adjustable by a potentiometer P may be impressed on the grid of the vacuum tube. A resistance and an ammeter are inserted in the filament circuit. The e.m.f. E_B is adjustable and an ammeter is provided to record the direct current in the plate circuit. The receiver may be connected to the plate circuit by a transformer T which may be shunted by a small capacity.

The set of Fig. 110 is complete and the different constants may be varied without large interactions. It may be simplified as conditions warrant. Thus, if but a limited range of wave lengths is to be covered, the loading L_0, C_1, C_2, L'_0 and the condenser C' may be omitted. If extreme sharpness of tuning is not required the variometer L may be omitted. The e.m.f.'s E_C and E_B may be given fixed values corresponding to the average requirements of the tubes to be used with the set. The feed back coil N'' may be coupled directly to L'_1 and the coil N' thus omitted. The e.m.f. E_C may be omitted entirely or a series grid condenser may be substituted for it.

Multiplex Telegraphy.—It will be remembered that the sharpness of the resonance curve obtained by a wave meter is limited by the sum of two decrements, namely that of the impressed e.m.f. and that of its own circuit. The same conditions hold of

course for a receiver and a transmitter. Sharpness of tuning and hence the possibility of discriminating at the receiver between two impressed wave trains is therefore greater for sustained waves than for damped waves. This increased ability to discriminate between wave trains of different frequencies means not only greater freedom from interference than was possible in the earlier days of the art, when damped waves were more generally used, but also increased possibilities for the selective, and hence useful, reception simultaneously of two or more wave trains of different frequency. Antenna circuits for such multiplex receiving have been described in the literature of the art

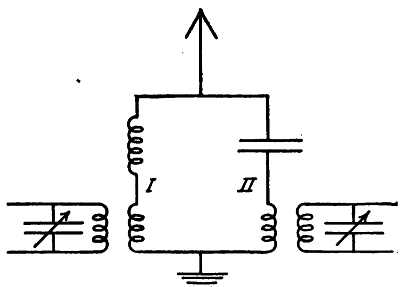


FIG. 111.—Antenna for multiplex telegraphy.

from time to time. The basic principle seems to have been the connection to the antenna of two receiving sets in parallel and the adjustment of these to different wave lengths. A typical form is shown in Fig. 111 where branches I and II of the antenna are tuned to different frequencies and coupled to independent receiving circuits. Similar antenna circuits for multiplex transmission have also been suggested.

In multiplex systems of the type described above electromagnetic waves of different frequencies are simultaneously transmitted from a single antenna and received by a single antenna. The discrimination or selectivity is accomplished at radio-frequency. Systems in which this selectivity is accom-

plished at audio-frequency have also been suggested. In such a system the transmitting antenna is excited by two transmitting systems using the same high frequency but having different tone frequencies. The reception is then accomplished with a single antenna and a single detector system but the output of the detector is selectively received in two tuned audio-frequency circuits.[1]

It is evident that the branched antenna of Fig. 111 may also be used for duplex telegraphy, that is for simultaneous operation as a receiving and a transmitting antenna, by replacing one of the receiving sets by a transmitter. The difficulties in the way of duplex telegraphy lie largely in the enormous differences between the energy in a transmitting circuit and that in a receiving circuit. Thus for long distance operation, *e.g.*, transatlantic, the transmitting circuit must carry a hundred or more amperes while the receiving circuit may be carrying a current so small that after detection it must be amplified to be audible in the receiver. The coupling or interaction of the transmitting branch on the receiving branch must therefore be reduced to an essentially impractical minimum. The method usually adopted therefore is to use separate transmitting and receiving antennæ, to locate them a few miles apart, to extend circuits from the relays in the transmitting station to keys in the receiving station and thus to operate by "remote control." Under these conditions the receiving antenna is usually lower and of cheaper construction so that the total cost of the aerial plant is much less than twice that of the transmitting antenna.

Secrecy Systems.—Applications of the principles involved in multiplex telegraphy have been suggested from time to time as a solution of the disadvantage in wireless transmission that the signals transmitted are essentially "broadcasted." Methods of this sort, involving the use either simultaneously or in prescribed orders of two or more frequencies, where the prescribed

[1] As for example in the experiments using "tone transmitters" carried out in 1910 by the Telefunken Co.

order is not in itself in the nature of a code, need only delay the success of an eavesdropper equipped with proper apparatus. The actual destruction of intelligibility, where such destruction is controlled at the sending end and where the intelligibility is reconstructed by apparatus at the receiving end, is open to the same objection, namely, that what one piece of apparatus will do for the legitimate listener a similar piece will accomplish for the eavesdropper. In times of war, however, or for commercial messages, a combination of such a mechanical system and a predetermined code for variations may easily be made effective for short messages. Even if an enemy station was provided with identical apparatus the time required before the proper adjustment was obtained might suffice and be of vital importance.

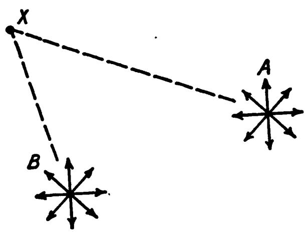


FIG. 112.—To determine position of a moving receiver.

Direction Finding.—The method by which groups of amateurs sometimes locate a new station approximately by comparing the intensity of the received signals at their several separate stations is simple and practical within limits. It assumes, of course, equal efficiencies of reception and a uniform attenuation of the observed signal in its transmission in different directions. Conversely a given station might obtain a very approximate idea of its own location by comparing the intensities of the signals received from several similar stations of known location. More accurate determinations may be made if each of the latter stations is so arranged as to transmit most efficiently in a given direction. Thus consider the stations indicated in Fig. 112 where X is a station of unknown position, *e.g.*, a boat or aeroplane. The stations of known location are grouped at a single point A but are directive in their transmission. Hence if they send one after the other in a prescribed rotation the line of direction from A to X will correspond to the direction of transmission of the

station whose signals are loudest at X. If a second group of directive transmitters are at B the location of X is of course determined by simple triangulation.

Directive Transmitters.—The simplest form of directive transmitter consists of two antennæ A_1 and A_2 separated by a distance d as in Fig. 113. The currents in the two antennæ are from the same source but are out of phase. Let the current in A_1 be $I\epsilon^{j(\omega t)}$ and that in A_2 be $I\epsilon^{j(\omega t+\theta)}$. At some point p_1 distant X along the line A_1A_2 the effect[1] of antenna A_1 will be $KI\epsilon^{j\left(\omega t+\frac{2\pi X}{\lambda}\right)}$ and that of antenna A_2 will be $KI\epsilon^{j\left(\omega t+\theta+\frac{2\pi[X+d]}{\lambda}\right)}$. Let the phase θ and the distance d be so chosen that at p_1 the

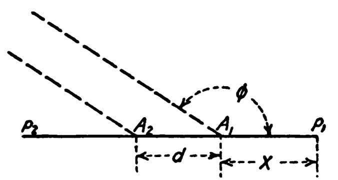


FIG. 113.—Directive transmitting with two antennæ.

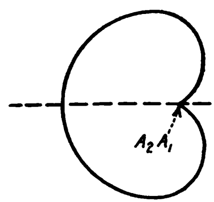


FIG. 114.—Typical direction-intensity characteristic for system of Fig. 113.

effects of the two antennæ neutralize. Then a difference in phase of π must exist between the two effects. Hence

$$\pm\pi + \left(\omega t + \frac{2\pi X}{\lambda}\right) = \left(\omega t + \theta + \frac{2\pi(X+d)}{\lambda}\right)$$

and

$$\pm\pi = \theta + \frac{2\pi d}{\lambda} \text{ or } \theta = \pi \pm \frac{2\pi d}{\lambda}$$

Along the line A_1p_1 there will then be no transmission. Substitutions of this value of θ and of negative values of X, gives the effect in the opposite direction.

For any other direction the effect of each antenna may then be obtained in terms of the angle φ, shown in the figure, by writing $(X + d\cos\varphi)$ for $(X + d)$ in the above expressions.

[1] In this connection see the development of the wave equation of p. 112.

If for each value of the angle φ there is plotted the resultant effect of the two antennæ a curve of the general form of Fig. 114 is obtained. The direction corresponding to the maximum is not necessarily A_1p_2 but depends upon the value of $\frac{d}{\lambda}$. This general method may be applied to determine the directive effect of any combination of antennæ.

Bellini-Tosi Directive System.—In the system of Bellini and Tosi two inclined vertical antennæ A_1 and A_2 as in Fig. 115 are excited from a common tuned source S of such frequency that the currents in the two verticals are 180° out of phase. To provide for varying the direction of the maximum transmission, which is in the plane of the two antennæ, the inventors provide a second pair of antennæ the plane of which is at right angles to that of the first pair. The coupling coil M of one pair is then at right angles to that of the other pair. The coupling coil N of the source is arranged to rotate so that in one position it will have zero coupling with one pair of antennæ and a maximum with the other pair. In rotating this coil through 90° the excitation of the two directive systems is varied from a maximum for one system (through a 45° position of equal excitation) to a maximum for the other system. The direction of the electro-magnetic disturbance transmitted from the combined systems is thus variable at will.

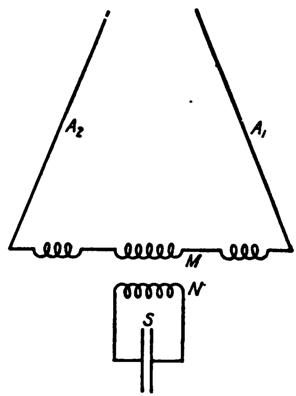


FIG. 115.—Bellini-Tosi direction system.

The apparatus is similarly applied to directive receiving in which case the rotating coil is connected to a receiving set. The position of the coil when the received signals are loudest will then indicate the line of direction of the distant source. In this form the system constitutes a "radio-goniometer."

Marconi Directive System.—The Marconi Company uses an antenna of the inverted L type with a horizontal portion of length one or two-fifths of the wave length transmitted. With this they find that the transmission is greatest in the direction opposite to that in which the horizontal portion extends away from the vertical portion. The relations for receiving and transmitting are reciprocal so that if two stations are established with the horizontal antenna at each station pointing away from the other station then they are in the best position for the interchange of signals.

Ground Antenna.—There has been some use made of long low antennæ which are practically long horizontal portions laid out along the ground. In general the receiving apparatus is located at the middle point. Such antennæ sometimes give surprisingly strong signals although they are not in general as efficient as the usual forms with high verticals. The electric force acting on an antenna was seen on page 114 to be vertical at the earth's surface, that is normal to the conducting plane formed by the earth's surface. In the case of a dry or poorly conducting layer on the surface of the earth the really effective conducting layer may be well below the surface at the location of the receiving antenna and inclined to it. With reference to that layer the horizontal ground antenna would be an inclined vertical. The action and possibilities of ground antennæ cannot therefore be foretold. The problem they present awaits further data. For the purposes of very temporary installations of receiving equipment they offer the advantage of simple construction and low cost and for military purposes the additional advantage of "low visibility." By the use of amplifiers the signals may be easily increased in intensity to any desired value.

Atmospheric Disturbances.—One of the most serious problems confronting the wireless engineer is that of reducing the effect of the interference at a receiving station of so-called "static" or atmospheric disturbances. The two general methods of increasing the discrimination between signals and static are (1)

to increase the output of the transmitting station and thus to increase the intensity of the received signals in comparison with the received static and (2) to design the receiving apparatus so that it will be selective to the transmitted signals as against the static. Along the line of the second method there have appeared from time to time in the literature of the art descriptions of devices and circuit arrangements for such a discriminatory selection.

To the student who is familiar with Chapter V it is evident that any e.m.f. active in an antenna will produce a forced current and a transient current. The former will depend for its value upon the impedance which the circuit offers to an e.m.f. of the wave form of that impressed, and the latter will depend upon the frequency constants of the antenna and upon the value of the forced current.

In a given antenna there is active at any instant an e.m.f. which is the sum or resultant of all those electromagnetic disturbances, occurring anywhere in the universe, which in their propagation through the ether have reached the antenna at the instant considered. In so far as this resultant e.m.f. from instant to instant undergoes variations periodic with the frequency of the antenna system it will give rise to a forced current of the same frequency as the transmitted wave trains. Since the causes of these static disturbances are infinite in number the e.m.f. impressed by the static may from time to time have characteristics of periodicities extending over a wide range of frequencies. The sharper the tuning the more limited the range of frequencies which will produce maximum effects in the antenna and hence in general the greater discrimination against static.

Of the various methods which have as yet been published none seem to have stood the test of time. So far as appears from periodical literature the most efficient method seems to be the simplest, namely, sharp tuning and loose coupling, the formation of signals by trains of continuous waves and their heterodyne detection in an efficient vacuum tube circuit.

APPENDIX

TRANSMISSION OVER WIRE CIRCUITS

Transmission over Wire Circuits.—Transmission over wire circuits whether telephonic or telegraphic may be considered as a special case of the general problem of the transmission of electromagnetic disturbances. In the case of transmission over wires the propagation of the disturbance created at the sending end is guided and restricted by the wires. Instead, therefore, of the disturbance being attenuated by spreading out in all directions from the source as well as by dissipation of energy in the medium between the two stations as is the case in wireless transmission, the attenuation in wire transmission is dependent solely upon the character of the wire circuits. As we shall see, the attenuation is easily predetermined when the type of circuit is known.

Waves in Wire Circuits.—An expression was developed in Chapter VII for the disturbance in the ether at a distance X from the source, when the source is an alternating-current generator. This expression is $KA \sin (\omega t - X/V)$ where ω is 2π times the frequency, V is the velocity with which disturbances are transmitted in the ether, A is the maximum amplitude of the disturbance and K is the attenuation factor. In the case of wireless transmission as has been noted this factor depends in a complicated manner on the distance, the wave length, the atmospheric and ground conditions. It can be determined only by a series of measurements for each set of conditions as was done by Austin for the case of transmission over sea water.

For the case of a wire circuit K is $\epsilon^{-\alpha x}$ where α is a constant for each type of line and for each given frequency. That this is so may be seen by considering the pure resistance line of Fig. 116.

This line is composed of a wire OO' between which and the return wire MM' there are uniformily spaced leakage paths. The wire OO' is composed of small resistances, r, in series. The wire MM', which may be the ground, is taken for simplicity as of negligible resistance. The leakage paths, which may be thought of as existing at the insulators in the case of an actual line, have each a resistance of r'. Now it is evident that of the current i flowing in one of the series elements, r, a certain part will be diverted by the next shunt element or leakage path. Hence the current flowing in any section of the line OO' is always less than that in the preceding section. If the line is assumed to be essentially infinite in length this attenuation of the current will result in a zero value of current at the infinitely distant terminal. For this length of line the distant terminal apparatus, since it gets

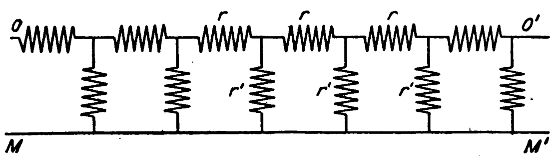


FIG. 116.—Transmission line with lumped resistance and conductance.

no current, can have no effect on the distribution of current among the series and shunt elements, r and r' respectively. Since these elements are alike for all sections, it follows that each shunt element will divert the same fractional part of the current flowing in the preceding series element. The current then in any series element is always the same fractional part of the current in the preceding series element. Let this fraction be q and count sections from the sending end. If the current in the first section has an amplitude of I, that in the second section will have an amplitude of qI, that in the third section of q^2I, *i.e.* $q^{3-1}I$. Hence the current in the $(n + 1)$ section will be q^nI.

Since q is a fraction less than unity it may be represented as $\epsilon^{-\alpha}$. If the length of line in a section is taken as unity, that is if the distance x from the first section to the section under con-

11

sideration is n (*e.g.* n miles if each section of line is 1 mile) then the amplitude of the current may be written

$$(\epsilon^{-\alpha})^n I = \epsilon^{-\alpha n} I = \epsilon^{-\alpha x} I.$$

The expression for the current i_x at a point distant x from the sending end may then be written as $i_x = I\epsilon^{-\alpha x} \sin (\omega t - \beta x)$, as will be seen by substituting in the expression for the propagation of a periodic disturbance the value of KA just obtained and also writing β for $1/V$. The constants α and β may be determined in terms of the impedances of the series and shunt elements of the line as will be seen later.

For the moment attention should be directed to the uses of the above expression. Upon substitution of any desired value of x the expression shows the manner in which the current at that point varies as time progresses. Similarly, upon substitution of any desired value of t, the expression shows how the currents which are flowing at that particular instant at various points along the line depend upon the position of these points. Thus let t have such a value that ωt is some multiple of 2π, then

$$\sin (\omega t - \beta x) = \sin (-\beta x) = -\sin \beta x$$

and the current at a distance x is

$$i_x = -I\epsilon^{-\alpha x} \sin \beta x \tag{1}$$

This is the equation of a damped sinusoid similar to the full line curve of Fig. 73 of page 112 except for the minus sign (that means for Fig. 73 A is negative).

The dotted curve of Fig. 73 represents the various currents at an instant later by about one-twelfth of a period of the alternating source. For the dotted curve the relation is then

$$i'_x = -I\epsilon^{-\alpha x} \sin (\pi/6 - \beta x) \tag{2}$$

Exponential Expression for Wave Motion.—On page 63 in connection with Fig. 36 it was seen that an expression of the form $A_0\epsilon^{-at}\epsilon^{j\omega t}$ represents a sinusoidal current which damps down as time progresses. In that case the damped sinusoid

shown on the right-hand side of Fig. 36 is plotted against time and the constantly decreasing vector shown on the left rotates with an angular speed of ω radians per second. The rotating vector makes one complete revolution in a cycle of the plotted sine curve. It is also possible to represent the damped sinusoid shown by the full line of Fig. 73 by a rotating and decreasing vector provided that the vector makes one revolution for each cycle of the sine curve. In this case the vector rotates through an angle which increases as the distance x increases. For the equation of the full line curve it is possible to write then

$$i_x = I\epsilon^{-\alpha x}\epsilon^{-j\beta x} \tag{3}$$

and for the dotted curve

$$i'_x = I\epsilon^{-\alpha x}\epsilon^{-j(\beta x - \pi/6)} \tag{4}$$

Of the two equations above, (3) is for the instant of time when ωt is some multiple of 2π and (4) is for the case when ωt is $\pi/6$ greater than for the preceding case. In general then it is possible to represent the current at a point distant x from the sending end by the expression

$$i_x = I\epsilon^{-\alpha x}\epsilon^{-j(\beta x - \omega t)}$$

or

$$i_x = I\epsilon^{-\alpha x - j\beta x}\epsilon^{j\omega t} \tag{5}$$

This expression for the disturbance propagated from an alternating source will be found to be perfectly general and especially convenient when rates of change are under discussion.

Space Rates and Time Rates.—Up to this point we have been concerned only with those variations in current or e.m.f. which as time progresses occur at a particular point in space. It is obvious, however, that in discussing the propagation of a periodic disturbance we are concerned with values of current and e.m.f. that vary as we turn our attention from one point to another along the line of propagation. Just as the symbol p was used to indicate the "time rate of change of" the quantity to which it is prefixed, so we may use the symbol D to represent the "space

rate of change" or in other words the rate at which the quantity to which D is prefixed varies with respect to space, *i.e.*, distance.

Consider now the current

$$i_x = I\epsilon^{-(\alpha + j\beta)x}\epsilon^{j\omega t} \tag{5'}$$

then

$$\begin{aligned} pi_x &= I\epsilon^{-(\alpha + j\beta)x}p\epsilon^{j\omega t} \\ &= I\epsilon^{-(\alpha + j\beta)x}(j\omega\epsilon^{j\omega t}) \\ &= j\omega i_x \end{aligned} \tag{6}$$

and

$$\begin{aligned} Di_x &= (I\epsilon^{j\omega t})D\epsilon^{-(\alpha+j\beta)x} \\ &= (I\epsilon^{j\omega t})(-[\alpha + j\beta]\epsilon^{-(\alpha+j\beta)x}) \\ &= -(\alpha + j\beta)i_x \end{aligned} \tag{7}$$

Lumped and Uniformly Distributed Lines.—In the line of Fig. 116 the series and shunt elements were treated as though the resistance and leakage of each section were localized or lumped into the resistances r and r'. In actual lines, however, these resistances are distributed, that is each small length of line, no matter how small, has some resistance and between it and the other wire there is some leakage. Also each little length of wire has some inductance and between it and the other wire there is some capacity. In actual lines, therefore, although the resistance and inductance may be and usually are expressed as a number of ohms and henries per mile it is necessary to remember that these quantities are uniformly distributed along each mile of length.

As to the capacity, it is evident that if between 1 inch of one wire and the other wire there is a certain capacity, then for 2 inches of wire there will be twice the capacity. The distributed capacity also may be expressed as a certain number of farads per mile. As to the leakage resistance, however, a different procedure is necessary since two leakage paths in parallel will offer only half the resistance that each alone would. It is convenient then to use the idea of conductivity as defined in Chapter I. Thus the leakage current across one path is the product of the voltage and the conductivity. The conductivity of two such paths is twice that of a single path since for the same voltage the

current would be twice as much. It is usual therefore in describing a wire circuit to express the conductivity as so many "mhos" per mile and to consider that it is uniformly distributed. Of course, in actual lines the conductivity is largely located at the insulators, but since there may be forty or more such points per mile and since the conductivity is in general small, the assumption of uniform distribution is sufficiently exact. In cable circuits the conductivity is actually uniformly distributed but is usually negligibly small.

When the circuit is formed by two wires, instead of by one wire and the ground, it is usual to express the constants of the circuit per loop (*i.e.*, per circuit) mile.

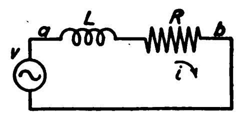


FIG. 117.—Series circuit of one section of line of Fig. 118.

Space Rates for Uniformly Distributed Lines.—Consider the series circuit of Fig. 117. The e.m.f. required to force a current i through the circuit is iZ where Z is the impedance. For the case shown Z is symbolically equal to $(R + Lp)$ and hence $v = Zi = (R + Lp)i$. In passing through this impedance from a to b there is a drop in voltage equal of course to Zi or $(R + Lp)i$. This is the change in voltage in going from a to b.

Now consider the circuit of Fig. 118 which represents a wire circuit. In going from a to b there is a change in voltage of $(R + Lp)i$. Similarly in going from a' to b' the change in voltage is $(R + Lp)i'$. In general in passing through the impedance of each series element of the line there is a change in voltage of Zi or $(R + Lp)i$ where R and L are the constants per section of the line, and i is the current in the section.

Just as the time rate of change of a voltage means the change in voltage per unit of time, so the space rate of change of voltage means the change per unit of length. Let the unit of length be the length of a section of the line, then in the series portion of the line the space rate of change of voltage, *i.e.* Dv, is Zi, in general,

or is $(R + Lp)i$ for the type of line shown in Fig. 118.[1] Of course if the circuit is one of uniformly distributed constants then the actual geographical length of each section is infinitesimally small.

Considering still Fig. 118 it is evident that in going from b to a' or from b' to a'' there is a change in current due to the leakage conductance G and the condenser C. The change in current per section, that is the rate of change of current, Di, is then the

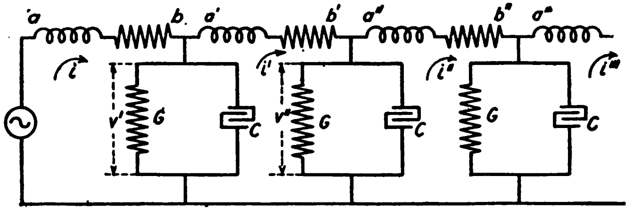


FIG. 118.—Transmission line with lumped L, R, C, and G.

current which passes through this shunt path. The current passing through the leakage path is $v'G$ or $v''G$ or of the general form vG. Similarly the current through the condenser is Cpv. The total change in current is then $(G + Cp)v$.

At any point therefore of an uniformly distributed circuit, as represented in Fig. 119, the voltage v across the circuit and the current i flowing in the circuit are related by the following equations:

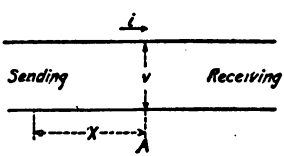


FIG. 119.—Uniformly distributed line.

$$Dv = -Zi = -(R + Lp)i \quad (8)^1$$
$$Di = -Z'v = -(G + Cp)v \quad (9)^1$$

The current i and the voltage v at A have been propagated from the sending end and hence are of the form given by equation (5). Thus

$$i = I\epsilon^{-(\alpha+j\beta)x}\epsilon^{j\omega t} \quad (5'')$$

$$v = E\epsilon^{-(\alpha+j\beta)x}\epsilon^{j\omega t} \quad (10)$$

[1] If increase is considered positive, then decrease is negative Since the voltage and current decrease, as x increases, a minus sign should be inserted as in equations (8) and (9).

where x is the distance from the sending end of the line and E and I are the maximum amplitudes of the voltage and current respectively and, hence, are the maximum voltage and current outputs of the alternating source at the sending end.

Impedance of a Transmission Line.—Finding the rates of change entering into equations (8) and (9) after the manner of equations (6) and (7) and substituting we have

$$-(\alpha + j\beta)v = -Zi = -(R + jL\omega)i \tag{8'}$$

$$-(\alpha + j\beta)i = -Z'v = -(G + jC\omega)v \tag{9'}$$

From these equations we may now determine v/i, that is the ratio, at any point of an infinitely long circuit, of the voltage impressed across the line and the resulting current. Thus divide equation (8′) by (9′) giving

$$\frac{v}{i} = \frac{Zi}{Z'v} \quad \text{or} \quad \frac{v^2}{i^2} = \frac{Z}{Z'}$$

whence

$$Z_0 = \frac{v}{i} = \sqrt{\frac{Z}{Z'}} = \sqrt{\frac{(R + jL\omega)}{(G + jC\omega)}} \tag{11}$$

It is important to note that this impedance is the same for all points of the line. It is therefore called the "characteristic impedance" or preferably the "iterative impedance." It is also evident that since R, L, G, and C enter as a fraction, it is immaterial in determining this impedance for what common unit of length they are specified.

Propagation Constant of Transmission Line.—In obtaining equation (11) we eliminated $(\alpha + j\beta)$. To determine this quantity, which is called the propagation constant of the line, we multiply equations (8′) and (9′) giving

$$(\alpha + j\beta)^2 = ZZ' = (R + jL\omega)(G + jC\omega)$$

whence

$$\gamma = (\alpha + j\beta) = \sqrt{ZZ'} = \sqrt{(R + jL\omega)(G + jC\omega)} \tag{12}$$

It is to be noted that in equations (5″) and (10) the propagation

constant is multiplied by the distance x, hence the line constants R, L, G, and C must be for the same unit of length as is used in stating this distance. To determine the components of γ, namely, α the attenuation constant and β the velocity constant, we write

$$(\alpha + j\beta)^2 = \alpha^2 + 2j\alpha\beta - \beta^2 = RG - LC\omega^2 + j(LG\omega + RC\omega)$$

hence

$$\alpha^2 - \beta^2 = RG - LC\omega^2 \tag{13}$$

and

$$2\alpha\beta = (LG + RC)\omega \tag{14}$$

From equations (13) and (14) both α and β may be determined. Certain special cases will now be considered.

Transmission in Twisted-pair Cable Circuits.—For cable circuits the leakage G is negligible and the inductance is also practically negligible since the two wires of the circuit are twisted together so that the inductance field of one wire practically neutralizes that of the other. Hence putting $L = O$ and $G = O$ in equations (13) and (14) gives

$$\alpha^2 - \beta^2 = 0 \tag{13'}$$

$$2\alpha\beta = RC\omega \tag{14'}$$

whence

$$\alpha^2 - \left(\frac{RC\omega}{2\alpha}\right)^2 = 0$$

or

$$4\alpha^4 = R^2C^2\omega^2$$

$$\alpha = \pm\sqrt{\frac{RC\omega}{2}} \tag{15}$$

Of these two values for α only the positive value has a physical significance since in $\epsilon^{-\alpha x}$ a positive value of α means an attenuation or reduction of available energy with distance. Solving for β gives $\beta = \alpha$. Hence for this case

$$\gamma = (1 + j1)\sqrt{\frac{RC\omega}{2}} \tag{16}$$

Equivalent Circuits.—The iterative impedance and the propagation constant of a distributed line both depend upon the frequency of the impressed disturbance. For any given frequency it is possible to replace any desired length of the distributed line by properly selected lumped impedances. Thus let it be desired to replace a length d by a T-network as shown in Fig. 120. For the frequency assumed this network must be equivalent in impedance and in propagation to the length d of the line which it replaces. Let the values of the impedances forming the net-work be Z_a and Z_b as shown and let the voltages and currents at A and B be v_1, i_1 and v_2, i_2 respectively. Then the conditions to be met in selecting Z_a and Z_b are

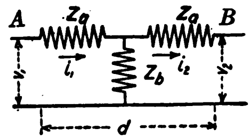


FIG. 120.—T-equivalent of a section of Fig. 118.

$$\frac{v_2}{v_1} = \frac{i_2}{i_1} = \epsilon^{-\gamma d} \tag{17}$$

$$\frac{v_1}{i_1} = Z_0 \tag{18}$$

To solve for Z_a and Z_b write the equations for the voltage in each branch in the same manner as for Fig. 41 page 75. Then

$$v_1 - i_1 Z_a - i_1 Z_b + i_2 Z_b = 0 \tag{19}$$

$$v_2 - i_1 Z_b + i_2 Z_a + i_2 Z_b = 0 \tag{20}$$

Substituting from equations (17) and (18) gives

$$v_1 = i_1[Z_a + Z_b(1 - \epsilon^{-\gamma d})] \tag{21}$$

$$v_2 = \epsilon^{-\gamma d} v_1 = i_1[Z_b(1 - \epsilon^{-\gamma d}) - Z_a \epsilon^{-\gamma d}] \tag{22}$$

$$Z_0 = Z_a + Z_b(1 - \epsilon^{-\gamma d}) \qquad \text{(23) from (21)}$$

$$Z_0 = -Z_a - Z_b(1 - \epsilon^{\gamma d}) \qquad \text{(24) from (22)}$$

$$Z_o^2 = Z_a^2 + 2Z_a Z_b \tag{25}$$

or

$$Z_0 = \frac{Z_b(\epsilon^{\gamma d} - \epsilon^{-\gamma d})}{2} \tag{26}$$

Also

$$Z_a = -Z_l + Z_b \frac{\epsilon^{\gamma d} + \epsilon^{-\gamma d}}{2} \tag{27}$$

$$Z_a = Z_0 \frac{\epsilon^{\frac{\gamma d}{2}} - \epsilon^{\frac{-\gamma d}{2}}}{\epsilon^{\frac{\gamma d}{2}} + \epsilon^{\frac{-\gamma d}{2}}} \tag{28}$$

Distributed Circuits with Lumped Loading.—The convenience of the T-network just discussed is evident when it is desired to

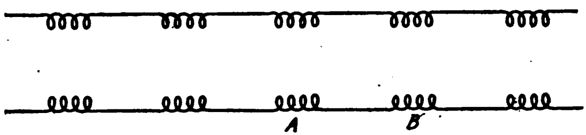


FIG. 121.—Uniformly distributed line with lumped loading.

find the iterative impedance and propagation constant for a circuit formed by a distributed line, in series with which, at regular intervals, additional impedances have been connected as shown in Fig. 121. Thus consider the distributed circuit between two successive "loads" to be replaced by its equivalent T. If this is done for the entire line as in Fig. 122, it is seen that the section included between A and B is a recurring section which also forms a symmetrical T. Let Z_c be the impedance of each load.

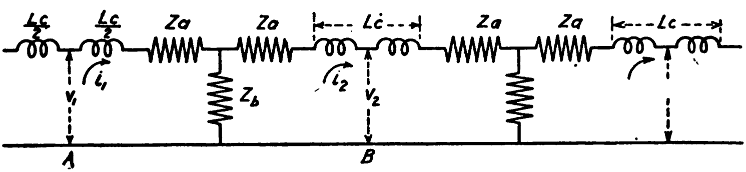


FIG. 122.—Line of Fig. 121 with T-equivalents for sections of distributed line.

For convenience put $Z_c = 2Z_c'$. The voltage equations for this section are similar to those of equations (19) and (20) and are

$$v_1 - i_1(Z_c' + Z_a + Z_b) + i_2 Z_l = 0 \tag{29}$$

$$v_2 - i_1 Z_b + i_2(Z_c' + Z_a + Z_l) = 0 \tag{30}$$

From which it is desired to find γ' and Z_0' where

$$\epsilon^{-\gamma' d} = \frac{v_2}{v_1} = \frac{\dot{i}_2}{\dot{i}_1} \tag{31}$$

and

$$Z_0' = \frac{v_1}{\dot{i}_1} \tag{32}$$

Substitute from (31) in (29) and (30) and solve for $\epsilon^{+\gamma' d}$ and also for $\epsilon^{-\gamma' d}$ whence

$$\epsilon^{\gamma' d} + \epsilon^{-\gamma' d} = 2\left(\frac{Z_a + Z_b + Z_c'}{Z_b}\right) \tag{33}$$

Substituting in (33) for Z_a as given by equation (27) gives

$$\frac{\epsilon^{\gamma' d} + \epsilon^{-\gamma' d}}{2} = \frac{\epsilon^{\gamma d} + \epsilon^{-\gamma d}}{2} + \frac{Z_c'}{Z_b} \tag{34}$$

Substituting from equation (31) in equations (29) and (30) and solving for $\frac{v_1}{\dot{i}_1}$ gives

$$Z_0'^2 = (Z_c' + Z_a)^2 + 2(Z_c' + Z_a)Z_b \tag{35}$$

Hyperbolic Functions.—It will be noticed that many of the equations obtained above involve terms of the form $\epsilon^{\gamma d} + \epsilon^{-\gamma d}$. The propagation constant γ is of the form $\alpha + j\beta$. For simplicity put d equal to unity and consider an expression of the form $\epsilon^{\gamma} + \epsilon^{-\gamma}$. Substituting $\gamma = \alpha + j\beta$ we have

$$\epsilon^{\gamma} = \epsilon^{\alpha + j\beta} = \epsilon^{\alpha}\,(\cos\beta + j\sin\beta) \tag{36}$$

$$\epsilon^{-\gamma} = \epsilon^{-\alpha - j\beta} = \epsilon^{-\alpha}(\cos\beta - j\sin\beta) \tag{37}$$

Hence

$$\epsilon^{\gamma} + \epsilon^{-\gamma} = (\epsilon^{\alpha} + \epsilon^{-\alpha})\cos\beta + j\,(\epsilon^{\alpha} - \epsilon^{-\alpha})\sin\beta \tag{38}$$

It will be remembered that

$$\cos\beta = \frac{\epsilon^{j\beta} + \epsilon^{-j\beta}}{2} \text{ and } j\sin\beta = \frac{\epsilon^{j\beta} - \epsilon^{-j\beta}}{2} \tag{39}$$

These expressions are exponential functions of an imaginary angle, namely $j\beta$. Without going into the mathematical deriva-

tion of these expressions for the sine and cosine we have made use of them in Chapter I.

In the same way we shall now find it convenient for simplicity of notation and for convenience in numerical examples to introduce two definitions from that branch of trigonometry which deals with hyperbolas instead of the circles with which the reader is already familiar. Thus the hyperbolic sine of an angle x is abbreviated as "sinh x" and is defined as

$$\sinh x = \frac{\epsilon^x - \epsilon^{-x}}{2} \tag{40}$$

Similarly the hyperbolic cosine of x, written "cosh x" is defined as

$$\cosh x = \frac{\epsilon^x + \epsilon^{-x}}{2} \tag{41}$$

Equation (38) may then be written as

$$\begin{aligned} \epsilon^\gamma + \epsilon^{-\gamma} &= 2 \cosh \gamma = 2 \cosh (\alpha + j\beta) \\ &= 2[\cosh \alpha \cos \beta + j \sinh \alpha \sin \beta] \end{aligned} \tag{42}$$

Some of the formulæ of the preceding pages may then be rewritten as follows:

$$Z_0 = Z_b \sinh \gamma d \tag{26'}$$

$$Z_a = - Z_b + Z_b \cosh \gamma d \tag{27'}$$

$$Z_a = Z_o \frac{\sinh \frac{\gamma d}{2}}{\cosh \frac{\gamma d}{2}} = Z_a \tanh \frac{\gamma d}{2} \tag{28'}$$

$$\cosh \gamma' d = \cosh \gamma d + \frac{Z_c}{2Z_b} \tag{34'}$$

Tables of the hyperbolic functions are in use similar in general form to those for the circular functions. The operation of calculating the value of γ' from equation (34') is one of finding the complex angle $\gamma' d$ whose hyperbolic cosine is another complex quantity, namely, cosh $(\gamma d + Z_c/2Z_b)$.

Propagation of an Alternating Current along a Circuit of Finite Length.—So far this discussion of circuits has been confined to the case of a circuit of essentially infinite length where the distant terminal apparatus receives zero current and zero voltage and hence the current and voltage relations along the line are conditioned solely by the line constants and the voltage impressed at the sending end.

Consider now the case of a circuit of iterative impedance Z_0 of propagation constant γ, and of length L which is terminated by impedances Z_r and Z_s at the receiving and sending ends respectively.

Let the impedance of the circuit at the sending end, that is the driving-point impedance, be represented by Z_1, and let the e.m.f. $E\epsilon^{j\omega t}$ be active in the impedance Z_s. At some point distant x from the sending end the voltage is

$$v = A\epsilon^{\gamma x} + B\epsilon^{-\gamma x} \qquad (43)^1$$

where A and B are constants to be determined by the terminal conditions. To find the current consider equation (8) namely

$$Dv = -Zi \qquad (8)$$

and substitute Dv as found from (43) giving

$$\gamma A\epsilon^{\gamma x} - \gamma B\epsilon^{-\gamma x} = -Zi \qquad (34)$$

or

$$i = -\frac{\gamma}{Z}(A\epsilon^{\gamma x} - B\epsilon^{-\gamma x})$$

or since

$$Z_0 = Z/\gamma$$

$$iZ_0 = -A\epsilon^{\gamma x} + B\epsilon^{-\gamma x} \qquad (44)$$

Equations (43) and (44) may be recast into forms involving $(A + B)$ and $(A - B)$. Thus consider equation (43)

[1] This is the complete solution for v in equations (8) and (9) of page 166.

$$v = A\epsilon^{\gamma x} + B\epsilon^{-\gamma x}$$

$$= \left(\frac{A}{2} + \frac{A}{2}\right)\epsilon^{\gamma x} + \left(\frac{B}{2} + \frac{B}{2}\right)\epsilon^{-\gamma x} + \left(\frac{A}{2} - \frac{A}{2}\right)\epsilon^{-\gamma x} + \left(\frac{B}{2} - \frac{B}{2}\right)\epsilon^{\gamma x}$$

$$= (B + A)\frac{(\epsilon^{\gamma x} + \epsilon^{-\gamma x})}{2} - (B - A)\frac{(\epsilon^{\gamma x} - \epsilon^{-\gamma x})}{2}$$

$$= (B + A) \cosh \gamma x - (B - A) \sinh \gamma x \qquad (43')$$

Similarly from equation (44)

$$iZ_0 = (B - A) \cosh \gamma x - (B + A) \sinh \gamma x \qquad (44')$$

To determine the constants $(A + B)$ and $(A - B)$ represent by v_1 the voltage impressed on the circuit at the sending end and by v_2 the voltage at the receiving end which is impressed on Z_r.

$$v_1 = E\epsilon^{j\omega t} - i_1Z_s \qquad (45)$$

$$v_2 = i_2Z_r \qquad (46)$$

Let $x = 0$ in equations (43) and (44) and substitute v_1 and i_1 for v and i.

$$v_1 = A + B \qquad (47)$$

$$i_1Z_0 = -A + B \qquad (48)$$

Hence equations (43′) and (44′) become

$$v = v_1 \cosh \gamma x - i_1Z_0 \sinh x \qquad (49)$$

$$iZ_0 = i_1Z_0 \cosh \gamma x - v_1 \sinh \gamma x \qquad (50)$$

Now in (49) and (50) let $x = L$ and substitute v_2 from (46)

$$v_2 = i_2Z_r = v_1 \cosh L\gamma - i_1Z_0 \sinh L\gamma \qquad (51)$$

$$i_2Z_0 = i_1Z_0 \cosh L\gamma - v_1 \sinh L\gamma \qquad (52)$$

From these two equations we obtain

$$Z_1 = \frac{v_1}{i_1} = \frac{Z_0(Z_0 \sinh L\gamma + Z_r \cosh L\gamma)}{(Z_r \sinh L\gamma + Z_0 \cosh L\gamma)} \qquad (53)$$

By equation (45) $v_1 = i_1 Z_1 = E\epsilon^{j\omega t} - i_1 Z_s$
or

$$i_1 = \frac{E\epsilon^{j\omega t}}{Z_s + Z_1}$$

$$= \frac{E\epsilon^{j\omega t}}{Z_s + Z_0 \dfrac{Z_r + Z_0 \tanh L\gamma}{Z_0 + Z_r \tanh L\gamma}} \tag{54}$$

By substitutions from (53) and (54) in (52) and by substituting $\cosh^2 L\gamma - \sinh^2 L\gamma = 1$ we obtain

$$i_2 = \frac{E\epsilon^{j\omega t}}{\left(Z_0 + \dfrac{Z_s Z_r}{Z_0}\right) \sinh L\gamma + (Z_s + Z_r) \cosh L\gamma} \tag{55}$$

If $Z_s = Z_r = Z_0$ then equations (53), (54) and (55) become

$$Z_1 = Z_0 \tag{53'}$$

$$i_1 = \frac{E\epsilon^{j\omega t}}{2Z_0} \tag{54'}$$

$$i_2 = \frac{E\epsilon^{j\omega t}}{2Z_0 (\sinh L\gamma + \cosh L\gamma)}$$

$$= \frac{E\epsilon^{j\omega t} \epsilon^{-L\gamma}}{2Z_0} = \frac{E\epsilon^{j\omega t} \epsilon^{-L\alpha} \epsilon^{-j\beta L}}{2Z_0} \tag{55'}$$

PROBLEMS

Part I. Graded Exercises

1. Multiplication and Division of Exponentials.

(*a*)	Multiply $3^2 \times 3^3$	*Solution.* $3^2 \times 3^3 = 3^{2+3} = 3^5$,
	Multiply $a^2 \times a^4$	*Solution.* $a^2 \times a^4 = a^{2+4} = a^6$.
	Multiply $a^n \times a^m$	*Ans.* a^{n+m}.
	Multiply $\epsilon^a \times \epsilon^y$	*Ans.* ϵ^{a+y}.
	Multiply $\epsilon^{j\theta} \times \epsilon^a$	*Ans.* $\epsilon^{j\theta+a}$.
	Multiply $\epsilon^{j\omega t} \times \epsilon^{j\theta}$	*Ans.* $\epsilon^{j(\omega t+\theta)}$.
	Multiply $\epsilon^{j(\omega t+\theta)} \times \epsilon^{-j\omega t}$	*Solution.* $\epsilon^{j\omega t-j\omega t+j\theta} = \epsilon^{j\theta}$.
(*b*)	What is 3^{-1}	*Ans.* $\frac{1}{3}$.
	What is 3^{-2}	*Ans.* $\frac{1}{3^2} = \left(\frac{1}{3}\right)^2$.
	What is a^{-n}	*Ans.* $\frac{1}{a^n}$.
	What is $3^4 \times 3^{-2}$	*Solution.* $3^{4-2} = 3^2$.
	What is $\frac{a^4}{a^2}$	*Solution.* $\frac{a^4}{a^2} = a^4 \times a^{-2} = a^{4-2} = a^2$.
	What is $a^m \times a^{-n}$	*Ans.* a^{m-n}.
	What is $a^k \times a^{-k}$	*Solution.* $a^{k-k} = a^0 = \frac{a^k}{a^k} = 1$.
	What is $(\epsilon^{j\omega t})(\epsilon^{-j\omega t})$	*Ans.* $\epsilon^0 = 1$.
(*c*)	What is $(2^2)^3$	*Solution.* $2^2 \times 2^2 \times 2^2 = 2^{3\times 2} = 2^6$.
	What is $(a^m)^n$	*Ans.* a^{mn}.
	What is $(\epsilon^{j\omega t})^2$	*Ans.* $\epsilon^{2j\omega t} = \epsilon^{j2\omega t}$.
	What is ϵ^0	*Ans.* 1.
	What is $4^{1/2}$	*Ans.* $\sqrt[2]{4}$.
	What is $4^{3/2}$	*Ans.* $\left(4^{1/2}\right)^3 = \left(\sqrt[2]{4}\right)^3$.
	What is $4^{-1/2}$	*Ans.* $\frac{1}{4^{1/2}} = \frac{1}{\sqrt{4}}$.
	What is $\left(4^{-1/2}\right)^3$	*Ans.* $4^{-3/2} = \left(\frac{1}{\sqrt{4}}\right)^3$.
		or $4^{-3/2} = (4^3)^{-1/2} = \frac{1}{\sqrt{4^3}}$.

What is $\epsilon^{4/5}$ *Ans.* $\sqrt[5]{\epsilon^4}$ or $(\sqrt[5]{\epsilon})^4$.

(*d*) What is $a^2 + a^3$ *Ans.* $a^2 + a^3$ or $a^2(1 + a)$.

What is $(a^2 + a^3)^2$ *Solution.* $(a^2)^2 + 2(a^2)(a^3) + (a^3)^2$
$= a^4 + 2a^5 + a^6$
or $[a^2(1 + a)]^2 = a^4[1 + 2a + a^2]$.

What is $\epsilon^{j\omega t} \times \epsilon^{-j(\omega t+\theta)}$ *Solution.* $\epsilon^{j\omega t}\epsilon^{-j\omega t}\epsilon^{-j\theta} = \epsilon^0\epsilon^{-j\theta} = \epsilon^{-j\theta}$.

What is $(\epsilon^{j\omega t} + \epsilon^{-j\omega t})^2$

Solution. $(\epsilon^{j\omega t})^2 + 2(\epsilon^{j\omega t})(\epsilon^{-j\omega t}) + (\epsilon^{-j\omega t})^2 = \epsilon^{j2\omega t} + 2 + \epsilon^{-j2\omega t}$.

(*e*) What is $A\epsilon^{j\omega t} \times B\epsilon^{j\omega t}$ *Ans.* $AB\epsilon^{j2\omega t}$.

What is $A\epsilon^{j\omega t} \times B\epsilon^{-j\omega t}$ *Ans.* AB.

What is $A\epsilon^{jxt} \times B\epsilon^{-jyt}$ *Ans.* $AB\epsilon^{j(x-y)t}$.

What is $10\epsilon^{j\omega t} \times 5\,\epsilon^{j\pi}$ *Ans.* $50\epsilon^{j(\omega t+\pi)}$.

What is $\dfrac{10\epsilon^{j(\omega t+\pi)}}{5\epsilon^{j\omega t}}$ *Solution.* $\dfrac{10}{5}\,\epsilon^{j(\omega t+\pi)\,-j\omega t} = 2\,\epsilon^{j\pi}$.

2. Resistance.

(*a*) If $v = 10$ volts and $i = 2$ amperes what is the resistance?

$$\textit{Solution. } R = \frac{10 \text{ volts}}{2 \text{ amp.}} = 5 \text{ ohms.}$$

(*b*) If 1.4 volts causes a current of 2 mil-amperes what is the resistance? NOTE.—1 mil-ampere = 0.001 ampere.

$$\textit{Solution. } R = \frac{v}{i} = \frac{1.4}{0.002} = 700 \text{ ohms.}$$

(*c*) What current will 6 volts cause in 3 megohms?
NOTE.—1 megohm $= 10^6$ ohms = 1,000,000 ohms.
1 micro-ampere $= 10^{-6}$ ampere = 0.000,001 ampere.

$$\textit{Solution. } i = \frac{v}{R} = \frac{6 \text{ volts}}{3 \times 10^6 \text{ ohms}} = \frac{2 \text{ amperes}}{10^6} = 2 \text{ micro-amperes.}$$

3. Conductivity.

(*a*) What is the conductivity of 1 ohm? *Ans.* 1 mho.
Of 1 megohm? *Ans.* 1 micro-mho.

(*b*) What is the current in a circuit of 4 mhos with 2 volts impressed?

Solution. $i = kv = 4 \times 2 = 8$ amperes.

For 2500 micro-mhos and 3 volts? *Ans.* 7500 micro-amperes.

(*c*) The e.m.f. v and the current i in a circuit are given by $i = 0 + 0.5v + 0.02\,v^2$. Find the current when 4 volts are impressed.

Solution. $i = 0 + 0.5\,(4) + 0.02\,(4^2)$, or $i = 2.32$ amps.

Find the resistance. *Solution.* $R = \frac{v}{i} = \frac{4}{2.32} = 1.72$ ohms.

Find the conductivity. *Ans.* $\frac{1}{1.72} = 0.58$ mho.

(*d*) In the circuit of 3-*c* what voltage is required to cause a current of 3 amperes?

Solution. Put $i = 3$ and solve $3 = 0.5v + 0.02v^2$ for v, thus $0.02v^2 + 0.5v - 3 = 0$

$$v = \frac{-0.5 + \sqrt{0.5^2 - 4(-3)(0.02)}}{2(0.02)} = +5 \text{ or } -30$$

The positive value of v representing a driving force in the same direction as the current is the required value.

4. Inductance.

(*a*) In an inductance the current at the moment under consideration is changing in value at such a rate that if it should continue to change (increase) at this rate for 1 sec. it would have increased 300 amperes. The e.m.f. active across the inductance at the given instant is 1.5 volts. What is the inductance?

Solution. $pi = 300 \frac{\text{amperes}}{\text{sec.}}$ $v = 1.5$ volts. $L = \frac{1.5 \text{ volts}}{300 \text{ amperes per sec.}} =$ 0.005 henry or 5 mil-henries.

(*b*) What is the value of the current at the instant under consideration.

Ans. The problem does not state and a knowledge of the value is not necessary to the foregoing solution.

(*c*) Is the current 300 amperes greater after 1 sec.?

Ans. The data given does not permit of an answer. If one estimates the speed of a passing automobile at 30 miles per hour it does not mean that at the end of 1 hr. it will be 30 miles away.

(*d*) Observations are made of the current in a circuit at two moments 0.001 sec. apart. In that time the current increases from 2 amperes to 3.4 amperes. What is the rate of change of the current?

Ans. The data is insufficient. If the current is an alternating one of 100,000 cycles it will have made 100 complete cycles in the time between the observations.

(*e*) If the current is an alternating current of 60 cycles frequency what is the rate of change?

Ans. The average rate of change is $\frac{3.4 - 2 \text{ amperes}}{0.001 \text{ sec.}} = 1400$ amperes per second. Since a 60 cycle current requires $\frac{1}{4 \times 60} = 0.00417$ sec. to change from its zero to its maximum value this average value of pi as determined for 0.001 sec. is quite close to the actual value.

(*f*) If the current in an inductance is of value $i = at$ and if the inductance is L henries what e.m.f. is required to maintain this current?

Solution. The current is always a times as large as the number of seconds since it started to flow. When t is zero the current is zero. But 1 sec. later when it is unity the current is a. At the end of each second it is a amperes larger than at the end of the preceding second. For 0.001 of a second it is (0.001 a) larger. In one-millionth of a second it increases one-millionth of a amperes. That is, its rate of change is uniform and equal to a amperes per second. Hence the e.m.f. required is $v = Lpi = La$ volts.

(*g*) If the current is $i = I + I_1at$ what is the required e.m.f.?

Solution. $pi = I_1a$. Hence $v = LI_1a$.

5. Capacity.

(*a*) At what rate is the e.m.f. increasing which is charging a condenser of 1 farad if a current of 1 ampere is flowing?

Solution. $C = \frac{i}{pv}$ or $pv = \frac{i}{C}$ $\therefore pv = \frac{1 \text{ amp.}}{1 \text{ farad}} = 1$ volt per second.

(*b*) If the capacity is 1 microfarad?

NOTE. 1 microfarad = 1 m.f. = 10^{-6} farads.

Solution. $pv = \frac{1 \text{ amp.}}{10^{-6} \text{ farad}} = 10^6$ volt/sec. = 1 mega-volt per second.

(*c*) If the capacity is 1 m.f. and the e.m.f. changes 1 volt/sec. what is the current?

Ans. 1 micro-ampere.

6. Angular Velocity.

(*a*) How many radians in 360°? *Ans.* 2π.

(*b*) What are the following in radians, 180°, 90°, 60°, 45°, 30°, A°.

Ans. π radians, $\frac{\pi}{2}$ radians, $\frac{\pi}{3}$ radians, $\frac{\pi}{4}$ radians, $\frac{\pi}{6}$ radians, $\frac{A\pi}{180}$ radians.

(*c*) If a flywheel is turning 1800 revolutions per minute what is its angular velocity in radians per second?

Solution. It turns 30 revolutions per second and each revolution is 2π radians, hence its angular velocity is $60\pi \frac{\text{radians}}{\text{sec.}}$.

(*d*) If $f = 50{,}000$ find ω. *Solution.* $\omega = 2\pi f = 314{,}159 \frac{\text{radians}}{\text{sec.}}$.

7. Trigonometric Ratios.

(*a*) In a right-angled triangle let one acute angle be θ radians, let the side adjacent to it be b inches long, the side opposite it a inches long. How long is c the hypothenuse? *Ans.* $c = \sqrt{a^2 + b^2}$ inches.

(*b*) What is $\sin \theta$ *Ans.* $\frac{a}{c}$.

What is $\cos\theta$ *Ans.* $\frac{b}{c}$.

What is $\tan\theta$ *Ans.* $\frac{a}{b}$.

(*c*) In what units is the sine measured?

Ans. It is a pure number, being merely the ratio of the number a (telling how many times the given unit of length goes into the side of a right triangle constructed on the angle) to the similar number c corresponding to the hypothenuse.

(*d*) What are the sines of $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$? What are the cosines of these angles?

Ans. $0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$, and $1, \frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}, 0$.

(*e*) Given $\theta = \frac{\pi}{6}$ and $c = 10$, find a and b.

Solution. $\frac{a}{c} = \sin\theta$ or $a = c\sin\theta = 10 \times \frac{1}{2} = 5$. Also $\frac{b}{c} = \cos\theta$ or $b = c\cos\theta = 10 \times \frac{\sqrt{3}}{2} = 8.66$.

(*f*) From the data of problem 7*d* find

$\theta = \tan^{-1}0$ *Ans.* Zero.

$\theta = \tan^{-1}\frac{1}{\sqrt{3}}$ *Ans.* $\frac{\pi}{6}$.

$\theta = \tan^{-1}\sqrt{3}$ *Ans.* $\frac{\pi}{3}$.

8. Vectors.

(*a*) If east is E and north is jE find expressions for NE, NW, SW and SE.

Solution.

$$\begin{aligned}
\overline{NE} &= E + jE \\
\overline{NW} &= j(\overline{NE}) = jE + j^2E = jE - E = -E + jE \\
\overline{SW} &= j^2(\overline{NE}) = -\overline{NE} = -E - jE \\
\overline{SE} &= j^3(\overline{NE}) = -j\overline{NE} = -jE - j^2E = E - jE
\end{aligned}$$

(*b*) Using $\sin 22°.5 = 0.383$ and $\cos 22°.5 = 0.924$ find expressions for $\overline{ENE}$, $\overline{NNE}$, etc.

Ans. $\overline{SSE} = 0.383E - j0.924E$, etc.

(*c*) Draw three vectors $6 + j0$, $3 + j4$, and $-2 + j2$. Find their sum by the parallelogram law. Check by adding as $(6 + j0) + (3 + j4) + (-2 + j2) = (6 + 3 - 2) + j(0 + 4 + 2) = 7 + j6$.

(*d*) Multiply $(3 + j4) \times (2 - j3)$.

Solution.

$$\begin{aligned}
3 \times 2 + j4 \times 2 + 3(-j3) + j4(-j3) &= 6 + j8 - j9 - j^2 12 \\
&= 6 - j1 + 12 = 18 - j1.
\end{aligned}$$

(*e*) Multiply $(a + jb) \times (a - jb)$. *Ans.* $a^2 + b^2$.

(*f*) Find the length of the vector $4 - j3$. *Solution.* $\sqrt{4^2 + 3^2} = 5$.

(*g*) Find the quotient $\dfrac{2 + j2}{4 + j3}$.

Solution. Multiply both numerator and denominator by $4 - j3$,

thus $$\frac{(2 + j2)(4 - j3)}{(4 + j3)(4 - j3)} = \frac{8 + 6 + j(8 - 6)}{4^2 + 3^2} = \frac{14 + j2}{25} = \frac{14}{25} + j\frac{2}{25}.$$

9. Vector Operator.

(*a*) Given $\sin 0 = 0$, $\cos 0 = 1$, $\sin \frac{\pi}{2} = 1$, $\cos \frac{\pi}{2} = 0$; find

$A\epsilon^{j0}$ *Ans.* A.

$A\epsilon^{j\frac{\pi}{2}}$ *Solution.* $\left(\cos \frac{\pi}{2} + j \sin \frac{\pi}{2}\right) A = jA$.

$A\epsilon^{j\left(\frac{\pi}{2} + \theta\right)}$ *Solution.* $A\epsilon^{j\frac{\pi}{2}}\epsilon^{j\theta} = jA\epsilon^{j\theta}$.

$A\epsilon^{j\pi}$ *Solution.* $A\epsilon^{j\pi} = A\epsilon^{j\frac{\pi}{2}}\epsilon^{j\frac{\pi}{2}} = j^2A = -A$.

$A\epsilon^{j\frac{3\pi}{2}}$ *Solution.* $A\epsilon^{j\frac{3\pi}{2}} = A\left(\epsilon^{j\frac{\pi}{2}}\right)^3 A\,(j)^3 = -jA$.

(*b*) Write the directions of the compass referred to $\overline{E}$.

Ans. $\overline{EbN} = \epsilon^{j\Phi}E$ where $\Phi = \frac{\pi}{16}$.

$$\overline{ENE} = \epsilon^{j\Phi}(\epsilon^{j\Phi})E = (\epsilon^{j\Phi})^2E = \epsilon^{j2\Phi}E$$

$$\overline{NEbE} = \epsilon^{j3\Phi}E$$

$$\overline{NE} = \epsilon^{j4\Phi}E, \text{ etc.}$$

$$\overline{N} = \epsilon^{j8\Phi}E = \epsilon^{j\frac{\pi}{2}}E = jE$$

$$\overline{WbN} = \epsilon^{j9\Phi}E = \epsilon^{j\left(\frac{\pi}{2} + \Phi\right)}E$$

$$\overline{NNW} = \epsilon^{j10\Phi}E = \epsilon^{j\left(\frac{\pi}{2} + 2\Phi\right)}E, \text{ etc.}$$

10. Sinusoidal Functions.

(*a*) What is $200\ \epsilon^{j628000\,t}$?

Ans. A vector of length 200 which revolves counterclockwise with an angular velocity of 628,000 radians per second.

(*b*) What are its component vectors?

Ans. $200 \cos 628{,}000\,t$ and $j\ 200 \sin 628{,}000\,t$.

(*c*) What is its frequency? *Ans.* $\dfrac{628000}{2\pi} = 100{,}000$ cycles/sec.

(*d*) What is its value at $t = 0$? *Ans.* 200.

(*e*) What is its value at $t = \frac{1}{4}\frac{1}{100000}$ sec.? *Ans.* $200\,\epsilon^{j\frac{\pi}{2}}$ or $j200$.

(*f*) What is its value at $t = \frac{1}{12}\frac{1}{100000}$ sec.?

Ans. $200\,\epsilon^{j\frac{\pi}{6}} = 200 \cos\frac{\pi}{6} + j200 \sin\frac{\pi}{6}$.

(*g*) If $200\,\epsilon^{j\omega t}$ is to represent a sinusoidal e.m.f. which starts at $t = 0$ from its maximum value of 200, which component is used?

Ans. The component $200 \cos \omega t$ which lies along the axis of reals?

(*h*) If the sinusoid is to be zero at $t = 0$?

Ans. The component along the axis of imaginaries or $j200 \sin \omega t$.

11. Phase.

(*a*) What is the phase difference between the rotating vectors (1) and (2)?

(1) $E\epsilon^{j\omega t}$ (2) $E\epsilon^{j(\omega t-\theta)}$ *Ans.* (1) leads (2) by θ.

(1) $E\epsilon^{j\omega t}$ (2) $E\epsilon^{j(\omega t+\pi)}$ *Ans.* (2) leads (1) by π.

(1) $E\epsilon^{j(\omega t+\theta)}$ (2) $E\epsilon^{j(\omega t-\theta)}$ *Ans.* (1) leads (2) by 2θ.

(1) $E\epsilon^{-j\omega t}$ (2) $E\epsilon^{-j(\omega t-\theta)}$

Ans. (1) is ahead of (2) by θ. Both vectors are rotating clockwise.

(1) $E\epsilon^{j\omega t}$ (2) $E\epsilon^{-j\omega t}$

Ans. The two vectors are rotating in opposite directions. We cannot describe the angular position of one of them with reference to that of the other by the use of a phase angle.

(*b*) What is the phase difference between the components of $E\epsilon^{j\omega t}$ and $E\epsilon^{-j\omega t}$?

Ans. For the real components, zero; for the imaginary components 180°.

(*c*) What is the phase difference corresponding to ½ cycle, 1 cycle, ¼ cycle, 2 cycles, ¾ cycle?

Ans. $\pi,\ 2\pi,\ \frac{\pi}{2},\ 4\pi,\ \frac{3\pi}{2}$.

12. General Expression for Vector.

(*a*) What is $E\epsilon^{j\omega t}$?

Ans. A vector of value E at $t = 0$ which rotates counterclockwise with an angular velocity ω.

(*b*) What is $(E_1 + jE_2)\epsilon^{j\omega t}$?

Ans. A vector of value $(E_1 + jE_2)$ at the instant $t = 0$. It rotates connterclockwise.

(*c*) If $E_2/E_1 = \tan\theta$ how may the vector of (*b*) be expressed?

Ans. $[\sqrt{E_1^2 + E_2^2}\,\epsilon^{j\theta}]\,\epsilon^{j\omega t} = \sqrt{E_1^2 + E_2^2}\,\epsilon^{j(\omega t+\theta)}$

(*d*) Find a vector 180° ahead of $(E_1 + jE_2)\epsilon^{j\omega t}$.

Ans. $\epsilon^{j\pi}[(E_1 + jE_2)\epsilon^{j\omega t}] = -(E_1 + jE_2)\epsilon^{j\omega t}$.

13. Conjugate Vectors.

(*a*) Write the conjugates of the following vectors.

$E\epsilon^{j\omega t}$ — *Ans.* $E\epsilon^{-j\omega t}$.

$E\epsilon^{j(\omega t+\theta)}$ — *Ans.* $E\epsilon^{-j(\omega t+\theta)}$.

$jE\epsilon^{j\omega t}$ — *Ans.* $-jE\epsilon^{-j\omega t}$.

$E_1 + jE_2$ — *Ans.* $E_1 - jE_2$.

$(E_1 + jE_2)\epsilon^{j\omega t}$ — *Ans.* $(E_1 - jE_2)\epsilon^{-j\omega t}$.

$(3 - j4)\epsilon^{-j(\omega t-\theta)}$ — *Ans.* $(3 + j4)\epsilon^{j(\omega t-\theta)}$.

(*b*) Write the sum of each of the following vectors and its conjugate.

$E\epsilon^{j\omega t}$ — *Ans.* $2E \cos \omega t$.

$jE\epsilon^{j\omega t}$ — *Ans.* $j^2 2E \sin \omega t = 2E \sin (\omega t + \pi)$.

$100\ (1 + j1)$ — *Ans.* 200.

$4 + j3$ — *Ans.* 8.

$E_1 + jE_2$ — *Ans.* $2E_1$.

$+ jE_1 - E_2$ — *Ans.* $-2E_2$.

$(E_1 + jE_2)\epsilon^{-j\omega t}$ — *Ans.* $2E_1 \cos \omega t$.

(*c*) What is $[(E_1 + jE_2)\epsilon^{j\omega t}][(E_1 - jE_2)\epsilon^{-j\omega t}]$? — *Ans.* $E_1^2 + E_2^2$.

14. Plot of $y = A\epsilon^{at}$.

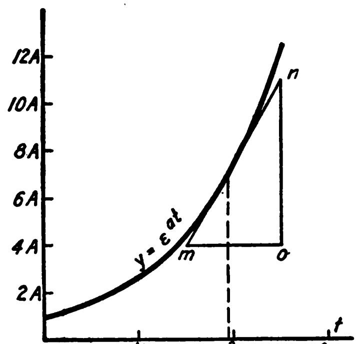


FIG. I.

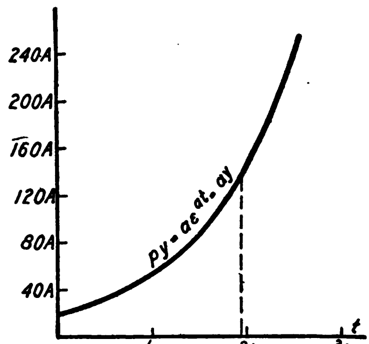


FIG. II.

Plot y as ordinates, and t as abscissæ as in Fig. I. Let 10 units of the cross-section paper represent A and 20 units represent $\frac{1}{20}$ second. For convenience put a equal to 20 so that when t is $\frac{1}{20}$ we have $at = 1$.

Solution. If $t = 0$, $at = 0$, $\epsilon^{at} = \epsilon^0 = 1$, and $y = A$

If $t = \frac{1}{20a}$, $at = \frac{1}{20}$ and from Table I we find $\epsilon^{-0.05} = \frac{1}{\epsilon^{-0.05}}$,

hence $y = \frac{A}{.9512} = 1.052A$. Similarly for other points.

15. Slope of $y = A\epsilon^{at}$.

Find slope of the curve of problem 14 and make a plot like that of Fig. II.

Solution. (Graphical) At any point as that in Fig. I, draw a tangent to the curve as mn. Draw mo and no parallel to the axes, then $\frac{no}{mo}$ is the slope. Plot this value as in Fig. II. It then appears that the slope is a curve of the same form as that for y, and is always equal to ay. Hence $py = ay = aA\epsilon^{at}$. Thus at $at = 1.95$, $y = 7.03A$. Also $\frac{no}{mo} = \frac{7.03A}{0.05} = 20\ (7.03)\ A = a\ y$.

16. To Find $p\epsilon^{j\omega t}$.

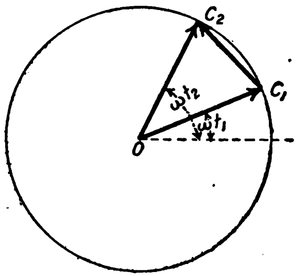


FIG. III.

Solution. Consider Fig. III where are shown two positions of a rotating vector, corresponding to times t_1 and t_2 respectively. Let these two values of the rotating vector be OC_1 and OC_2, respectively, where $OC_1 = \epsilon^{j\omega t_1}$ and $OC_2 = \epsilon^{j\omega t_2}$. The vector OC_2 is obviously equal to $OC_1 + C_1C_2$. That is the vector C_1C_2 represents the change in the quantity $\epsilon^{j\omega t_1}$ which occurs in the time interval $t_2 - t_1$. If this interval, $t_2 - t_1$, is small so that the angle ωt_2 of Fig. III is not much greater than the angle ωt_1 then C_1C_2 differs but slightly from the arc of the circle between C_1 and C_2. Now the angle $\omega t_2 - \omega t_1$ is measured by this arc, thus

$$\omega t_2 - \omega t_1 = \frac{\text{arc } C_1C_2}{OC_1}$$

Hence C_1C_2 which is the change in the vector OC_1 during the time interval, $t_2 - t_1$, is given by $C_1C_2 = jOC_1\,(\omega t_2 - \omega t_1) = j\omega\,(OC_1)\,(t_2 - t_1)$. The direction of C_1C_2 being 90° counter clockwise from OC_1 is indicated by the operator j.

The change per second, which is $p\epsilon^{j\omega t_1}$ is then

$$p\epsilon^{j\omega t_1} = \frac{C_1C_2}{t_2 - t_1} = j\omega(OC_1) = j\omega\epsilon^{j\omega t_1}.$$

In general, then, we may write $p\epsilon^{j\omega t} = j\omega\epsilon^{j\omega t}$.

17. To Find $p^{-1}y$. If $y = A\epsilon^{at}$ find $p^{-1}y$.

Solution. Let $Y = \dfrac{y}{a}$ then $pY = \dfrac{py}{a} = A\epsilon^{at}$. Now, $p^{-1}y$ means "the quantity whose rate of change is y". If $y = A\epsilon^{at}$ then $p^{-1}y$ equals $p^{-1}(A\epsilon^{at})$ and "means the quantity whose rate of change is $A\epsilon^{at}$," but this quantity is obviously Y since $pY = A\epsilon^{at}$. Hence $p^{-1}y = Y = \dfrac{y}{a} = \dfrac{A\epsilon^{at}}{a}$.

18. Write the rates of chenge for the following quantities.

$A\epsilon^{j\omega t}$ *Ans.* $j\omega(A\epsilon^{j\omega t})$.

$A\epsilon^{j(\omega t+\theta)}$

Solution. $A\epsilon^{j(\omega t+\theta)} = [A\epsilon^{j\theta}]\epsilon^{j\omega t} = A_0\epsilon^{j\omega t}$, hence $p(A_0\epsilon^{j\omega t}) = j\omega A_0\epsilon^{j\omega t}$
$= j\omega A\epsilon^{j\theta}\epsilon^{j\omega t} = j\omega A\epsilon^{j(\omega t+\theta)}$.

$E\epsilon^{j(\omega t+\theta)} + E\epsilon^{-j(\omega t+\theta)}$ *Ans.* $j\omega E\epsilon^{j(\omega t+\theta)} - j\omega E\epsilon^{-j(\omega t+\theta)}$

19. Write $p^{-1}y$ for the following.

$y = A\epsilon^{j\omega t}$ *Ans.* $\dfrac{y}{j\omega} = \dfrac{A\epsilon^{j\omega t}}{j\omega}$.

$y = A\epsilon^{-j\omega t}$ *Ans.* $\dfrac{y}{-j\omega} = \dfrac{A\epsilon^{-j\omega t}}{-j\omega}$.

$y = A\epsilon^{j\theta}\epsilon^{j\omega t}$ *Ans.* $\dfrac{y}{j\omega}$.

$y = A\epsilon^{(j\omega+a)t}$ *Ans.* $\dfrac{y}{j\omega + a}$.

20. If $i = I\epsilon^{j\omega t} + I\epsilon^{-j\omega t}$ where $I = 10$ amperes and $\omega = 377$, find the current when $t = \dfrac{2\pi}{12} \times \dfrac{1}{377}$ sec.

Solution. $\omega t = \dfrac{2\pi}{12} = \dfrac{\pi}{6}$ then $i = 10\epsilon^{j\frac{\pi}{6}} + 10\epsilon^{-j\frac{\pi}{6}}$ or $i = 2\left(10\cos\dfrac{\pi}{6}\right)$ $= 20(0.867) = 17.34$ amperes.

21. Find pi for the conditions of problem 20.

Solution. $pi = j\omega I\epsilon^{j\omega t} - j\omega I\epsilon^{j\omega t}$. Hence when $\omega t = \dfrac{\pi}{6}$:

$$pi = 377\left[[2(10)\sin\frac{\pi}{6}\right] = 3770 \text{ amperes per second}$$

22. Find $p^{-1}i$ for the conditions of problem 20.

Solution. $p^{-1}i = \dfrac{I\epsilon^{j\omega t}}{j\omega} + \dfrac{I\epsilon^{-j\omega t}}{-j\omega}$.

Substituting for ωt and ω gives

$$p^{-1}i = \frac{10\epsilon^{j\frac{\pi}{6}}}{j377} + \frac{10\epsilon^{-j\frac{\pi}{6}}}{-j377}$$

Multiplying numerator and denominator by $-j$ gives

$$p^{-1}i = \frac{-j10\epsilon^{j\frac{\pi}{6}}}{377} + \frac{j10\epsilon^{-j\frac{\pi}{6}}}{377}$$
$$= \frac{20}{377}\sin\frac{\pi}{6} = \frac{10}{377} \text{ ampere seconds.}$$

That is $\dfrac{10}{377}$ coulombs since 1 coulomb = 1 ampere × 1 second.

Part II. Circuits

NOTE.—For convenience only one vector of a pair of conjugates is used in those problems where the omission is allowable as explained on p. 22. For those problems where the complete expression is necessary, the word "conjugate" follows the problem number.

1. Find the impedance offered by an inductance L to a sinusoidal current of frequency f.

Solution. Assume the current flowing in the circuit to be $i = I\epsilon^{j\omega t}$ where ω is the angular velocity, namely $2\pi f$, and I is the maximum value of the current. From the definition of inductance, the e.m.f. required to force this current through the circuit is $v = Lpi$. Substituting for pi gives $v = j\omega Li$.

This means that the e.m.f. required to maintain the current is 90° ahead of the current and is $L\omega$ times as great. The impedance is $Z = \frac{v}{i} = j\omega L$.

2. If the circuit of problem 1 contains both a resistance R and the inductance L, what is the impedance?

Solution. The e.m.f. required across the resistance is Ri, that is, it is in phase with the current and R times as large. The e.m.f. required across the circuit is then the sum of that across the resistance and the value of problem 1.

Hence $$v = Ri + Lpi$$
or
$$v = Ri + jL\omega i.$$
Hence $$Z = \frac{v}{i} = R + jL\omega$$
or
$$Z = \sqrt{R^2 + L^2\omega^2}\,(\epsilon^{j\theta})$$
where $$\theta = \tan^{-1}\frac{L\omega}{R}.$$

3. If the circuit of problem 1 contains a condenser of capacity C instead of the inductance, find its impedance.

Solution. By definition $i = Cpv$,

hence $\frac{p^{-1}i}{C} = v$. Substituting $p^{-1}i = \frac{i}{j\omega}$ gives $v = \frac{i}{jC\omega}$ and $Z = \frac{1}{jC\omega} = \frac{-j}{C\omega}$.

In this case the required e.m.f. is 90° behind the current and of value $\frac{1}{C\omega}$ times the current.

4. If R, L, and C are in series, find the impedance.

Ans. $v = Ri + Lpi + \frac{p^{-1}i}{C} = \left(R + jL\omega - \frac{j}{C\omega}\right)i.$

Hence $$Z = R + j\left(L\omega - \frac{1}{C\omega}\right) = \sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}\left(\epsilon^{j\theta}\right)$$
where $$\theta = \tan^{-1}\frac{L\omega - \frac{1}{C\omega}}{R}.$$

5. (*a*) An alternating current of maximum value 10 amps. and of frequency 79,600 cycles per second flows in a resistance of 6 ohms, find the required e.m.f.

Solution. $i = 10\epsilon^{j2\pi 79600t} = 10\epsilon^{j5t \times 10^5}$.

Hence $$v = Ri = 6 \times 10\epsilon^{j5t \times 10^5} = 60\epsilon^{j5t \times 10^5}$$

(*b*) What is the value of this e.m.f. ¼ cycle after t is zero?

Solution. Put $\omega t = \frac{2\pi}{4} = \frac{\pi}{2}$. Then $v = 60\epsilon^{j\frac{\pi}{2}}$.

If in the expression $i = I\epsilon^{j\omega t}$ we are tacitly omitting the imaginary component, we must omit the corresponding component in this case giving the value of the e.m.f. when $\omega t = \frac{\pi}{2}$ as $v = 60 \cos \frac{\pi}{2} = 0$.

6. (*a*) If the current of problem 5 (*a*) flows through an inductance of 1 mil-henry, what is the required e.m.f.?

Solution. $L = 0.001$.

$$\begin{aligned} v &= jL\omega i = j\,(10^{-3})\,(2\pi 79600)(10)\epsilon^{j2\pi 79600t} \\ &= j5000\epsilon^{j5t \times 10^5}. \end{aligned}$$

(*b*) What is the voltage at the end of ¼ cycle?

Solution. Put $\omega t = \frac{\pi}{2}$, then $v = j5000\,\epsilon^{j\frac{\pi}{2}} = j5000\left(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}\right)$

or $$v = 5000\left(-\sin\frac{\pi}{2} + j\cos\frac{\pi}{2}\right)$$

and omitting the imaginary component as before gives -5000 volts.

7. (*a*) If the current of problem 5 (*a*) is the charging current of a condenser of 2000 micro-micro-farads, what is the required e.m.f.?

Solution.

$$v = \frac{i}{j\omega C} = \frac{i}{j(5 \times 10^5)\,(2000 \times 10^{-12})} = -j10^3 i = -j10000\epsilon^{j5t \times 10^5}$$

(*b*) What is the e.m.f. at the end of ¼ cycle? *Ans.* 10,000 volts.

8. (*a*) If the resistance, inductance and capacity of problems 5, 6 and 7 are all in series, what is the e.m.f. required across the circuit to cause the current of 5 (*a*) to flow?

Solution.

$$\begin{aligned} v &= \left(R + jL\omega - \frac{j}{C\omega}\right)i = (6 + j500 - j1{,}000)i \\ &= (60 - j5000)\epsilon^{j5t \times 10^5} \\ &= 5{,}000\epsilon^{j5t \times 10^5 + 0.012 - \frac{\pi}{2}} \end{aligned}$$

since $$\tan^{-1}\frac{5000}{60} = \frac{\pi}{2} - 0.012$$

9. (*a*) If two circuits, 1 and 2, are in series and if the e.m.f's. required to force a current of $i = I\epsilon^{j\omega t}$ through them are respectively

$$v_1 = E_1\epsilon^{j(\omega t+\theta_1)} = E_1\epsilon^{j\omega t}\cdot\epsilon^{j\theta_1} \qquad \text{and} \qquad v_2 = E_2\epsilon^{j(\omega t+\theta_2)} = E_2\epsilon^{j\omega t}\cdot\epsilon^{j\theta_2}$$

what is the e.m.f. required across the series circuit?

Solution. $v = v_1 + v_2 = \epsilon^{j\omega t}[E_1\epsilon^{j\theta_1} + E_2\epsilon^{j\theta_2}] = E_0\epsilon^{j\omega t}$

where $E_0 = E_1\epsilon^{j\theta_1} + E_2\epsilon^{j\theta_2}$

$$= [(E_1\cos\theta_1 + E_2\cos\theta_2) + j(E_1\sin\theta_1 + E_2\sin\theta_2)]$$

(*b*) If $E_1 = 100$, $E_2 = 80$, $\theta_1 = \frac{\pi}{6}$, $\theta_2 = \frac{\pi}{3}$, and $\omega = 5 \times 10^5$, find v

Ans. $v = (126.6 + j119.3)\epsilon^{j5t \times 10^5}$.

(*c*) Draw to scale the vectors of 9 (*b*) at the instant $t = 0$. Draw the vector E_0. Check the sum $E_1\epsilon^{j\theta_1} + E_2\epsilon^{j\theta_2}$ by completing the parallelogram.

10. (*a*) Two impedances $Z_1\epsilon^{j\theta_1}$ and $Z_2\epsilon^{j\theta_2}$ are in parallel. An e.m.f. of $E\epsilon^{j\omega t}$ is applied to the terminals of the branched circuit. Find the impedance of the branched circuit.

Solution. The total current i supplied to the circuit is the sum of the currents i_1 and i_2 supplied to the individual branches. The impedance of the branched circuit is the ratio of the e.m.f. to the total current. Hence,

$$i_1 = \frac{E\epsilon^{j\omega t}}{Z_1\epsilon^{j\theta_1}} = \frac{E}{Z_1}\epsilon^{j(\omega t-\theta_1)}$$

$$i_2 = \frac{E\epsilon^{j\omega t}}{Z_2\epsilon^{j\theta_2}} = \frac{E}{Z_2}\epsilon^{j(\omega t-\theta_2)}$$

$$i = E\epsilon^{j\omega t}\left(\frac{1}{Z_1\epsilon^{j\theta_1}} + \frac{1}{Z_2\epsilon^{j\theta_2}}\right)$$

$$\frac{v}{i} = \frac{Z_1Z_2\epsilon^{j(\theta_1+\theta_2)}}{Z_1\epsilon^{j\theta_1} + Z_2\epsilon^{j\theta_2}}$$

(*b*) If $3 + j4$ and $4 + j3$ are the two impedances, find the total impedance in ohms.

Solution. $\theta_1 = \tan^{-1}\frac{4}{3} = \frac{\pi}{2} - \theta_2 \qquad \text{since } \theta_2 = \tan^{-1}\frac{3}{4}$

$Z_1\epsilon^{j\theta_1} + Z_2\epsilon^{j\theta_2} = 3 + j4 + 4 + j3 = 7 + j7$

$Z_1 = 5 \qquad Z_2 = 5.$

$$Z = \frac{5.5.\epsilon^{j\frac{\pi}{2}}}{7+j7} = \frac{25\epsilon^{j\frac{\pi}{2}}}{7\sqrt{2}\,\epsilon^{j\frac{\pi}{4}}} = 2.53\epsilon^{j\frac{\pi}{4}} = 1.79 + j1.79$$

(*c*) Find the total current if an e.m.f. of the frequency for which the impedances are given above and of maximum value 100 volts is impressed on the branched circuit.

Solution. $v = 100\epsilon^{j\omega t}$

$$i = \frac{100\epsilon^{j\omega t}}{2.53\epsilon^{j\frac{\pi}{4}}} = 39.6\epsilon^{j\left(\omega t-\frac{\pi}{4}\right)}$$

11. (*a*) In a series circuit $R = 3$ ohms, $L = 0.002h$, and $C = 0.001$ *mf*. Find the frequency for which the circuit is resonant.

$$\textit{Solution.}\ L\omega = \frac{1}{C\omega} \text{ or } \omega^2 = \frac{1}{LC} = \frac{1}{(2 \times 10^{-3})(1 \times 10^{-9})} = 50 \times 10^{10}$$

$$\omega = 7.07 \times 10^5,\ f = \frac{\omega}{2\pi} = \frac{7.07}{2\pi} \times 10^5 = 112{,}500 \text{ cycles per sec.}$$

(*b*) If an e.m.f. of 1.1 times the resonant frequency is impressed on this circuit, what is the impedance to it?

$$\textit{Solution.}\ Z = R + j\left(L\omega - \frac{1}{C\omega}\right) = 3 + j\frac{LC\omega^2 - 1}{C\omega}$$

Substitute $\omega = 1.1\omega_0 = \dfrac{1.1}{\sqrt{LC}}$. Hence $Z = 3 + j\dfrac{1.21 - 1}{1.1C\omega_0}$ or $Z = 3 + j266$.

(*c*) If the maximum value of the e.m.f. of 11 (*b*) is the same as that of an *e.m.f.* of the resonant frequency, find the ratio of the currents for the two e.m.f.'s.

$$\textit{Solution.}\ \frac{\text{current of resonant freq.}}{\text{current of 1.1 resonant freq.}} = \frac{\sqrt{3^2 + 266^2}}{3} = 89$$

12. (*a*) An inductance L and a condenser C are in parallel. Find the impedance of the circuit to an e.m.f. of frequency $\dfrac{1}{2\pi\sqrt{LC}}$

Solution. Using the results of problem 10*a* we obtain

$$Z = \frac{(jL\omega)\left(\frac{-j}{C\omega}\right)}{j\left(L\omega - \frac{1}{C\omega}\right)} = -j\frac{L\omega}{(LC\omega^2 - 1)} = \infty \text{ for } \omega^2 = \frac{1}{LC}$$

(*b*) If the resistance of the inductance is not zero but is R ohms, find the impedance.

$$\textit{Solution.}\ Z = \frac{(R + jL\omega)\left(\frac{-j}{C\omega}\right)}{R + j\left(L\omega - \frac{1}{C\omega}\right)}. \qquad \text{Placing } \omega^2 = \frac{1}{LC}$$

$$Z = \frac{-\frac{jR}{C\omega} + \frac{L}{C}}{R} = \frac{L}{CR} - j\frac{1}{C\omega} = \frac{L}{CR} - j\sqrt{\frac{L}{C}}$$

$$= \sqrt{\frac{L}{C}}\left(\frac{\sqrt{\frac{L}{C}}}{R} - j\right).$$

13. (Conjugate) Effective Value of a Sinusoid.—The effective value of an alternating current is defined as the value of a steady direct current which would produce in a resistance a heating effect equal to the average effect which the alternating current would produce. Since the instantaneous

heating effect of a current i is proportioned to i^2 this definition states that the effective value of an alternating current is its "root mean square value." Find the effective value of a sinusoidal current in terms of its maximum value.

Solution. Let the maximum value be $I_m = 2I$ and the current be $i = I\epsilon^{j\omega t} + I\epsilon^{-j\omega t}$.
Then $i^2 = I^2\epsilon^{j2\omega t} + I^2\epsilon^{-j2\omega t} + 2I^2\epsilon^{j0t}$ is evidently composed of a sinusoidal effect of double the frequency of the current and a zero frequency effect. The average of the double frequency sinusoid is of course zero, hence the average of i^2 is merely the constant effect $2I^2$. The effective value I_{eff} is the square root of this, hence

$$I_{eff} = \sqrt{2}I = \frac{\sqrt{2}I_m}{2} = \frac{I_m}{\sqrt{2}} = 0.707I_m$$

14. (Conjugate) Average Power.—The instantaneous value of the power expended in a circuit is vi where v is the e.m.f. impressed on the circuit and i is the resulting current. If the e.m.f. and current are sinusoidal find the average power in terms of the phase angle θ by which v leads i.

Solution. Let $v = E\epsilon^{j(\omega t + \theta)} + E\epsilon^{-j(\omega t+\theta)}$ and $i = I\epsilon^{j\omega t} + I\epsilon^{-j\omega t}$.
Then $vi = EI[\epsilon^{j(2\omega t+\theta)} + \epsilon^{-j(2\omega t+\theta)}] + EI[\epsilon^{j\theta} + \epsilon^{-j\theta}]$
In this expression for vi the first part represents a sinusoidal effect proportional to EI and of double the frequency. Its average will be zero. The second part represents an effect which does not vary with the time, that is, a steady effect, of value $EI\ [\epsilon^{j\theta} + \epsilon^{-j\theta}] = 2EI \cos \theta$.
Hence the power fluctuates with double frequency sinusoidally about the average value of $2EI \cos \theta$. Since $2E = E_m$ and $2I = I_m$ and $E_{eff} = \dfrac{E_m}{\sqrt{2}}$ and $I_{eff} = \dfrac{I_m}{\sqrt{2}}$, this average power may be written $E_{eff} I_{eff} \cos \theta$. The factor "$\cos \theta$" is called the "power factor."

(*b*) Solve by the usual methods of calculus the preceding problem.

Solution.
$$v = E_m \sin (\omega t + \theta)$$
$$i = I_m \sin \omega t$$
$$vi = E_m I_m \sin \omega t \sin (\omega t + \theta)$$

$$\text{Average power} = \frac{\int_0^{\frac{2\pi}{\omega}} vi\,dt}{\int_0^{\frac{2\pi}{\omega}} \omega\,dt}$$

15. A battery of E volts is connected to a circuit composed of an inductance L and resistance R in series. Find the expression for the current.

Solution. Let the current be $i = I_0 + I\epsilon^{(-a+j\omega)t}$ (1)
where I_0, I, a, and ω are to be determined. The e.m.f. across the circuit is

$$E = Ri + Lpi \tag{2}$$

Note that at the instant when the battery is connected the current is zero. That is, for $t = 0$, $i = 0$. Hence

$$i = I_0 + I\epsilon^{(-a+j\omega)0} = I_0 + I = 0 \quad \text{and } I = -I_0$$

Substitute $i = I_0 - I_0\epsilon^{(-a+j\omega)t}$ and $pi = -(-a + j\omega)I_0\epsilon^{(-a+j\omega)t}$ in equation (2) giving

$$E = RI_0 - RI_0\epsilon^{(-a+j\omega)t} - (-a + j\omega)LI_0\epsilon^{(-a+j\omega)t} \tag{3}$$

This relation must of course hold for all values of t, hence putting $t = 0$ gives $E = -(-a + j\omega)LI_0$ therefore $\omega = 0$ and $I_0 = \frac{E}{La}$.
Substituting for I_0 and ω in equation (3) gives

$$E = \frac{RE}{La} - \frac{RE}{La}\epsilon^{-at} + E\epsilon^{-at}$$

hence

$$a = \frac{R}{L}$$

and substituting in equation (1) gives

$$i = \frac{E}{R}\left(1 - \epsilon^{-\frac{Rt}{L}}\right)$$

This solution is much facilitated by the principles expressed on page 68. The forced current will obviously be E/R. The transient current at $t = 0$ will be equal and opposite to the forced current, hence the complete expression for the transient will be $-\frac{E}{R}\epsilon^{(-a+j\omega)t}$.
Since there is only one type of storage reservoir the current will not be oscillatory, hence ω will be zero. To find a put $Z = R + jLp = 0$ from which $p = -\frac{R}{L}$ or $-a = -\frac{R}{L}$

16. A condenser of capacity C which is charged to a voltage E is allowed to leak (*i.e.*, discharge) through a resistance R. Find the equation of the discharge current.

Solution. The impedance of the series circuit formed by the resistance and capacity is $R + \frac{p^{-1}}{C}$. Hence in the expression for the natural oscillation of the circuit namely $I\epsilon^{pt}$ we have $p = -\frac{1}{RC} + j0$. Represent the total current in the circuit by $I_0 + I\epsilon^{pt}$ where I_0 is the forced current.

The impressed e.m.f. is zero hence the forced current is zero and the total current is

$$i = I\epsilon^{\frac{-t}{RC}} \tag{1}$$

To find I write the general expression $v = Ri + \frac{p^{-1}i}{C}$ and note that v is always zero; that at $t = 0$ the e.m.f. across the condenser is E, hence at $t = 0$ we have $0 = Ri + E$ or $i_0 = -\frac{E}{R}$. But from (1) above when $t = 0$ we have $i_0 = I$ then $I = -\frac{E}{R}$ and the current in the circuit is $i = -\frac{E}{R}\epsilon^{\frac{-t}{RC}}$. The minus sign before E indicates, of course, that the current flows in the opposite direction to E, *i.e.*, is a discharge current.

17. In a damped wave train the decrement is 0.15. Find the number of whole waves before the amplitude is reduced to one-tenth of its initial value.

Solution. The wave train may be represented by $I\epsilon^{-at}\epsilon^{j\omega t}$.

For a substitute df. For t substitute $\frac{n}{f}$ where n is the required number of waves. Then $\epsilon^{-at} = \epsilon^{-nd}$. From Table I it is found that $\epsilon^{-2.30} = 0.1$. Hence we have $nd = 2.30$
or since

$d = 0.15$, the required number is $n = 15 +$

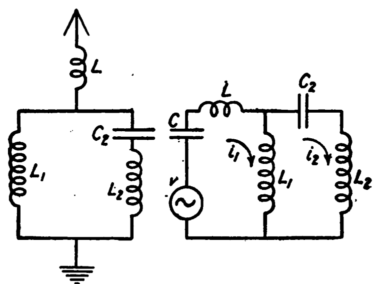


FIG. IV.

18. The coupling between the two circuits of Fig. 41, p. 75, is 0.20. The frequency of each circuit by itself is 30,000 cycles per second. Find the two natural frequencies of the coupled circuit. *Ans.*—27,400 and 33,500.

19. A branched antenna is shown in Fig. IV, also its equivalent circuit. The constants are $C = 0.001$ *mf.* $L = 0.00024h$, $L_1 = 0.0023h$, $C_2 = 0.001$ *mf.*, $L_2 = 0.0033\ h$.

Find the two natural frequencies with which it will oscillate.

Solution. To find the natural frequencies equate the driving-point impedance to zero. To find it assume the currents i_1 and i_2 as shown and write the equations for the e.m.f's. thus

$$v = \left(Lp + L_1p + \frac{1}{Cp}\right) i_1 - L_1pi_2$$

$$0 = -L_1pi_1 + \left(L_2p + \frac{1}{C_2p}\right) i_2$$

$$\text{hence } Z = \frac{v}{i_1} = \frac{\left(Lp + L_1p + \frac{1}{Cp}\right)\left(L_2p + \frac{1}{C_2p}\right) - L_1^2p^2}{L_2p + \frac{1}{C_2p}}$$

Equating the numerator to zero gives

$$(LL_2 + L_1L_2 - L_1^2)\,p^2 + \left(\frac{L_1}{C_2} + \frac{L}{C_2} + \frac{L_2}{C}\right) + \frac{1}{CC_2p^2} = 0$$

Multiplying through by CC_2p^2 and substituting the numerical values gives

$$3.09 \times 10^{-24}p^4 + 5.84 \times 10^{-12}p^2 + 1 = 0$$

hence $$p^2 = 10^{12}\,\frac{5.84 \pm \sqrt{34.1 - 12.36}}{6.18}$$

and $p^2 = -19.1 \times 10^{10}$ or -1.70×10^{12}

$\therefore \quad j\omega = j\,4.37 \times 10^5$ or $j\,13.04 \times 10^5$

$f = 69{,}600$ or $208{,}000$

$\lambda = 4310m.$, or $1440m.$

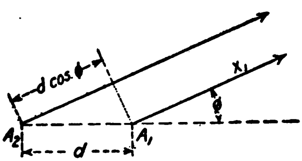


FIG. V.

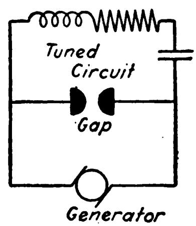


FIG. VI.

20. Find the effect at a distance x, large as compared to d, of the directive system described on p. 145.

Solution. Let the effect of A_1 be

$$KI\epsilon^{j\left(\omega t + \frac{2\pi x_1}{\lambda}\right)}$$

then that of A_2 is

$$KI\epsilon^{j\left(\omega t + \theta + \frac{2\pi x_2}{\lambda}\right)}$$

Put $\theta = -\pi + \frac{2\pi d}{\lambda}$ and $x_2 = x_1 + d\cos\varphi$ as follows from Fig. V. Then the total effect is $KI\epsilon^{j\omega t}\epsilon^{j\frac{2\pi x_1}{\lambda}}\left(1 + \epsilon^{-j\pi}\cdot\epsilon^{j\frac{2\pi d\,(1+\cos\phi)}{\lambda}}\right)$ or writing $y = \frac{2\pi d\,(1+\cos\varphi)}{\lambda}$ and $K_0 = KI\epsilon^{j\frac{2\pi x_1}{\lambda}}$. Then the effect is $K_0\epsilon^{j\omega t}(1 - \epsilon^{jy})$. The absolute value of this effect is then $2K_0\epsilon^{j\omega t}(1 - \cos y)$.

21. In the circuit of Fig. VI the generator voltage is 550. The spark gap is reduced until a discharge occurs. Assume the discharging gap to introduce a resistance of 4 ohms, independent of the current through it. The condenser is 0.001 mf. The inductance is 10 mil-henries and has a resistance of 10 ohms. Find the current in the tuned circuit following the break down of the gap, neglecting any current supplied from the generator.

Solution. Assume the current to be $i = (I_1 + jI_2)\epsilon^{pt} + (I_1 - jI_2)\epsilon^{\bar{p}t}$ where I_1, I_2, p and $\bar{p}$ are to be determined. Of these quantities p and $\bar{p}$ are found as on p. 73 to be $p = -\frac{R}{2L} + \frac{j}{\sqrt{LC}}\sqrt{1 - \frac{R^2C}{4L}}$ and its conjugate. Hence, upon substitution of numerical values $p = -700 + j3.16\times 10^5$. To find I_1 and I_2 write the equation for the e.m.f. in the circuit namely

$$v = Ri + Lp(I_1 + jI_2)\epsilon^{pt} + L\bar{p}(I_1 - jI_2)\epsilon^{\bar{p}t} + \frac{(I_1 + jI_2)\epsilon^{pt}}{pC} + \frac{(I_1 - jI_2)\epsilon^{\bar{p}t}}{\bar{p}C}$$

and note that at $t = 0$ the e.m.f. across the condenser is equal and opposite to that of the generator, say E, hence since v is zero after the spark discharges we have at $t = 0$

$$0 = 2RI_1 - 2LaI_1 - 2L\omega I_2 - E$$

Also note that at the instant $t = 0$ the current is zero for at this instant the conditions are the same as those of problem 15, where an e.m.f. is suddenly impressed on a circuit containing an inductance and resistance, only in this case the e.m.f. E is due to the condenser. Hence, at $t = 0$, $i = 0$ and $2I_1 = 0$ Therefore

$$I_2 = \frac{-E}{2L\omega} = \frac{-550}{2(10^{-2})(3.16\times 10^5)} = -\ 0.0872 \text{ ampere.}$$

Hence the current is $i = j0.0872\left(\epsilon^{(-700-j31600)t} - \epsilon^{(-700+j31600)t}\right)$ or $i = 0.174\,\epsilon^{-700t}\sin 31{,}600\,t$. The decrement is $\frac{a}{f}$ or 0.0138.

TABLE I.—VALUES OF ϵ^{-x}

x	ϵ^{-x}	x	ϵ^{-x}	x	ϵ^{-x}
0.000	1.0000	0.36	0.6977	0.82	0.4404
.005	0.9950	.37	.6907	.84	.4317
.010	.9900	.38	.6839	.86	.4232
.015	.9851	.39	.6771	.88	.4148
.020	.9802	.40	.6703	.90	.4066
.025	.9753	.41	.6637	.92	.3985
.030	.9704	.42	.6570	.94	.3906
.035	.9656	.43	.6505	.96	.3829
.040	.9608	.44	.6440	.98	.3753
.045	.9560	.45	.6376	1.00	.3679
.050	.9512	.46	.6313	1.05	.3499
.055	.9465	.47	.6250	1.10	.3329
.060	.9418	.48	.6188	1.15	.3166
.065	.9371	.49	.6126	1.20	.3012
.070	.9324	.50	.6065	1.25	.2865
075	.9277	.51	.6005	1.30	.2725
080	.9231	.52	.5945	1.35	.2592
085	.9185	.53	.5886	1.40	.2466
.090	.9139	.54	.5827	1.45	.2346
.095	.9094	.55	.5769	1.50	.2231
.100	.9048	.56	.5712	1.55	.2122
.110	.8958	.57	.5655	1.60	.2019
.12	.8869	.58	.5599	1.65	.1920
.13	.8781	.59	.5543	1.70	.1827
.14	.8694	.60	.5488	1.75	.1738
.15	.8607	.61	.5434	1.80	.1653
.16	.8521	.62	.5379	1.85	.1572
.17	.8437	.63	.5326	1.90	.1496
.18	.8353	.64	.5273	1.95	.1423
.19	.8270	.65	.5220	2.00	.1353
.20	.8187	.66	.5169	2.10	.1225
.21	.8106	.67	.5117	2.20	.1108
.22	.8025	.68	.5066	2.30	.1003
.23	.7945	.69	.5016	2.40	.0907
.24	.7866	.70	.4966	2.50	.0821
.25	.7788	.71	.4916	2.60	.0743
.26	.7711	.72	.4868	2.70	.0672
.27	.7634	.73	.4819	2.80	.0608
.28	.7558	.74	.4771	2.90	.0550
.29	.7483	.75	.4724	3.00	.0498
.30	.7408	.76	.4677		
.31	.7334	.77	.4630		
.32	.7261	.78	.4584		
.33	.7189	.79	.4538		
.34	.7118	.80	.4493		
.35	.7047				

NOTE TO TABLE II

NOTE.—Reading down the table gives sines and cosines for θ from 0 to $\frac{\pi}{4}$ by steps of 0.01 radian. For angles between $\frac{\pi}{4}$ and $\frac{\pi}{2}$ the sines and cosines may be found by the formulas

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

For convenience the table is arranged to read up to give the sines and cosines of angles greater than $\frac{\pi}{4}$. The steps are also by 0.01 radian but start from $\frac{3.14159}{2} - 0.78 = 0.7908$ radian. For the greater part of the table the figures 0.0008 may be neglected in which case the value of the trigonometric ratio will be correct to at least three significant figures.

TABLE II.—VALUES OF SIN θ AND COS θ FOR θ IN RADIANS

Read down θ radians	Read down θ degrees	sin θ	cos θ	Read up degrees	Read up radians
0.00	0° 0′.0	0.0000	1.0000	90° 0′	1.57(08)
.01	0°34′.4	.0100	.9999	89°25′.6	1.56(08)
.02	1° 8′.8	.0200	.9998	88°51′.2	1.55(08)
.03	1°43′.1	.0300	.9995	88°16′.9	1.54(08)
.04	2°17′.5	.0400	.9992	87°42′.5	1.53(08)
.05	2°51′.9	.0500	.9987	87° 8′.1	1.52(08)
.06	3°26′.3	.0600	.9982	86°33′.7	1.51(08)
.07	4° 0′.6	.0699	.9975	85°59′.4	1.50(08)
.08	4°35′.0	.0799	.9968	85°25′.0	1.49(08)
.09	5° 9′.4	.0899	.9959	84°50′.6	1.48(08)
.10	5°43′.8	.0998	.9950	84°16′.2	1.47(08)
.11	6°18′.2	.1098	.9940	83°41′.9	1.46(08)
.12	6°52′.5	.1197	.9928	83° 7′.5	1.45(08)
.13	7°26′.9	.1296	.9916	82°33′.1	1.44(08)
.14	8° 1′.3	.1395	.9902	81°58′.7	1.43(08)
.15	8°35′.6	.1494	.9888	81°24′.4	1.42(08)
.16	9°10′.0	.1593	.9872	80°50′.0	1.41(08)
.17	9°44′.4	.1692	.9856	80°15′.6	1.40(08)
.18	10°18′.8	.1790	.9838	79°41′.2	1.39(08)
.19	10°53 .2	.1889	.9820	79° 6′.8	1.38(08)
.20	11°27′.5	.1987	.9801	78°32′.5	1.37(08)
.21	12° 1′.9	.2085	.9780	77°58′.1	1.36(08)
.22	12°36′.3	.2182	.9759	77°23′.7	1.35(08)
.23	13°10′.7	.2280	.9737	76°49′.3	1.34(08)
.24	13°45′.1	.2377	.9713	76°14′.9	1.33(08)
.25	14°19′.4	.2474	.9689	75°40′.6	1.32(08)
.26	14°53′.8	.2571	.9664	75°6′.2	1.31(08)
.27	15°28′.2	.2667	.9638	74°31′.8	1.30(08)
.28	16° 2′.6	.2764	.9611	73°57′.4	1.29(08)
.29	16°36′.9	.2860	.9582	73°23′.1	1.28(08)
.30	17°11′.3	.2955	.9553	72°48′.7	1.27(08)
.31	17°45′.7	.3051	.9523	72°14′.3	1.26(08)
.32	18°20′.1	.3146	.9492	71°39′.9	1.25(08)
.33	18°54′.5	.3240	.9460	71° 5′.5	1.24(08)
.34	19°28′.8	.3335	.9427	70°31′.2	1.23(08)
.35	20° 3′.2	.3429	.9394	69°56′.8	1.22(08)
.36	20°37′.6	.3523	.9359	69°22′.4	1.21(08)
.37	21°11′.9	.3616	.9323	68°48′.0	1.20(08)
.38	21°46′.3	.3709	.9287	68°13′.7	1.19(08)
.39	22°20′.7	.3802	.9249	67°39′.3	1.18(08)
.40	22°55′.1	.3894	.9211	67° 4′.9	1.17(08)

TABLE II.—VALUES OF SIN θ AND COS θ FOR θ IN RADIANS.—(*Continued*)

θ radians	θ degrees	sin θ	cos θ	degrees	radians
.41	23°29′.5	.3986	.9171	66°30′.5	1.16(08)
.42	24° 3′.9	.4078	.9131	65°56′.1	1.15(08)
.43	24°38′.2	.4169	.9090	65°21′.8	1.14(08)
.44	25°12′.6	.4259	.9047	64°47′.4	1.13(08)
.45	25°47′.0	.4350	.9004	64°13′.0	1.12(08)
.46	26°21′.4	.4439	.8960	63°38′.6	1.11(08)
.47	26°55′.7	.4529	.8916	63° 4′.3	1.10(08)
.48	27°30′.1	.4618	.8870	62°29′.9	1.09(08)
.49	28° 4′.5	.4706	.8823	61°55′.4	1.08(08)
.50	28°38′.9	.4794	.8776	61°21′.1	1.07(08)
.51	29°13′.3	.4882	.8727	60°46′.7	1.06(08)
.52	29°47′.6	.4969	.8678	60°12′.4	1.05(08)
	30° 0′.0	.5000	.8667	60° 0′.0	
.53	30°22′.0	.5055	.8628	59°38′.0	1.04(08)
.54	30°56′.4	.5141	.8577	59° 3′.6	1.03(08)
.55	31°30′.8	.5227	.8525	58°29′.2	1.02(08)
.56	32° 5′.1	.5312	.8473	57°54′.9	1.01(08)
.57	32°39′.5	.5396	.8419	57°20′.5	1.00(08)
.58	33°13′.9	.5480	.8365	56°46′.1	.99(08)
.59	33°48′.3	.5564	.8309	56°11′.7	.98(08)
.60	34°22′.6	.5646	.8253	55°37′.4	.97(08)
.61	34°57′.0	.5729	.8196	55° 3′.0	.96(08)
.62	35°31′.4	.5810	.8139	54°28′.6	.95(08)
.63	36° 5′.8	.5891	.8080	53°54′.2	.94(08)
.64	36°40′.2	.5972	.8021	53°19′.8	.93(08)
.65	37°14′.5	.6052	.7961	52°45′.5	.92(08)
.66	37°48′.9	.6131	.7900	52°11′.1	.91(08)
.67	38°23′.3	.6210	.7838	51°36′.7	.90(08)
.68	38°57′.7	.6288	.7776	51° 2′.3	.89(08)
.69	39°32′.0	.6365	.7712	50°28′.0	.88(08)
.70	40° 6′.4	.6442	.7648	49°53′.6	.87(08)
.71	40°40′.8	.6518	.7584	49°19′.2	.86(08)
.72	41°15′.2	.6594	.7518	48°44′.8	.85(08)
.73	41°49′.6	.6669	.7452	48°10′.4	.84(08)
.74	42°23′.9	.6743	.7385	47°36′.1	.83(08)
.75	42°58′.3	.6816	.7317	47° 1′.7	.82(08)
.76	43°32′.7	.6889	.7248	46°27′.3	.81(08)
.77	44° 7′.0	.6961	.7179	45°52′.9	.80(08)
.78	44°41′.4	.7033	.7109	45°18′.6	.79(08)
	45° 0′	.7071	.7071	45° 0′	
		cos θ	sin θ	degrees	radians
				Read up	

TABLE III

Electrical Units (Practical)

NOTE.—Multiples and sub-multiples of the practical units are designated by prefixes with meanings as follows:

Mega.—Used to express a unit 10^6 times as large as that to which it is prefixed.

Kilo.—10^3 times.

Mil.—10^{-3} times, *i.e.*, one-thousandth.

Micro.—10^{-6} times, *i.e.*, one-millionth.

Micro-micro.—10^{-12} times, *i.e.*, one-millionth of one-millionth.

The practical unit of **current** is the *ampere.*

The practical unit of **e.m.f.** is the *volt.*

In terms of these the remaining units may be defined as follows:

Quantity.—*The coulomb.*—1 coulomb is the quantity of electricity transferred in 1 second by a current of 1 ampere.

Resistance.—*The ohm.*—1 ohm is the resistance of a circuit in which 1 ampere flows under a steady e.m.f. of 1 volt.

Inductance.—*The henry.*—1 henry is the inductance of a circuit which is linked by its own magnetic field when a change of current in it at the rate of 1 ampere per second establishes an e.m.f. of 1 volt.

Capacity.—*The farad.*—1 farad is the capacity of a condenser when a charging current of 1 ampere is maintained by an e.m.f. which is increasing at the rate of 1 volt per second.

Energy.—*The joule.*—1 joule is the energy expended in a circuit in 1 second by a current of 1 ampere under an e.m.f. of 1 volt.

Power.—*The watt.*—1 watt is the power in a circuit where energy is expended at the rate of 1 joule per second.

Angular Velocity.—*Radian per second.*

Impedance.—*The Ohm.*—NOTE.—The product of an inductance and an angular velocity is an impedance in ohms. The reciprocal of the product of a capacity and an angular velocity is an impedance in ohms.

Admittance.—*The mho.*—A circuit of impedance 1 ohm has an admittance of 1 mho. NOTE.—Admittance in mhos is the reciprocal of impedance in ohms.

Reactance.—*The ohm.*—That portion of the impedance which represents the impedance in an equivalent series circuit of the reservoir for energy storage.

Conductance.—*The mho.*—That portion of the admittance which represents in an equivalent branched circuit, one branch of which is resistance and the other reactance, the reciprocal of the resistance.

Susceptance.—*The mho.*—The reciprocal of the reactance of the equivalent branched circuit described under "conductance" above.

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